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Some Operations over Pythagorean Fuzzy Matrices Based on Hamacher Operations

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Abstract

A Pythagorean fuzzy matrix is a powerful tool for describing the vague concepts more precisely. The Pythagorean fuzzy matrix based models provide more flexibility in handling the human judgment information as compared to other fuzzy models. The objective of this paper is to apply the concept of intuitionistic fuzzy matrices to Pythagorean fuzzy matrices. In this paper, we briefly introduce the Pythagorean fuzzy matrices and some theorems and examples are applied to illustrate the performance of the proposed methods. Then we define the Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations on Pythagorean fuzzy matrices and investigate their algebraic properties. Furthermore, we prove some properties of necessity and possibility operators on Pythagorean fuzzy matrices.

Keywords: Pythagorean fuzzy matrix; Hamacher operations; Multiplication; Exponentiation

MSC 2010 No.: 03E72, 15B15, 94D05

1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. (2002) and simultaneously Im et al. (2001) to generalize the concept of Thomason (1977) fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle a_{ij}, a'_{ij} \rangle$. The sum $a_{ij} + a'_{ij}$ of each ordered pair is less than or equal to 1. Since the appearance of IFM, several researchers have importantly contributed to the development of IFM theory and its applications, resulting in the greater success from the theoretical and technological points of view. Further, Emam and Fndh (2016) defined some kinds of IFMs. Also, they construct an idempotent IFM from any given one through the min-max composition. Pal (2001) introduced the intuitionistic fuzzy determinant and (2006) defined some basic operations and relations of IFMs including maxmin, minmax, complement, algebraic sum, algebraic product, etc. and proved equality between IFMs. Mondal and Pal (2013) studied the similarity relations, together with invertibility conditions and eigenvalues of IFMs. Zhang and Xu (2012) studied intuitionistic fuzzy value and introduced the concept of composition two intuitionistic fuzzy matrices. Muthuraji et al. (2016) obtained a decomposition of intuitionistic fuzzy matrices. Sriram and Boobalan (2016) studied the properties of algebraic sum and algebraic product of intuitionistic fuzzy matrices and prove that the set of all intuitionistic fuzzy matrices form a commutative monoid.

Yager (2013) introduced the concept of the Pythagorean fuzzy set (PFS) and developed some aggregation operations for PFS. Yager, in 2014, the PFS characterized by a membership degree and a nonmembership degree satisfying the condition that the square sum of its membership degree and nonmembership degree is equal to or less than 1, has much stronger ability than IFS to model such uncertain information in MCDM problems. Zhang and Xu (2014) studied various binary operations over PFS and also proposed a decision making algorithm based on PFS. Peng and Yang (2015) developed a Pythagorean fuzzy superiority and inferiority ranking method to solve uncertainty multiple attribute group decision-making problem. Using the theory of PFS, we defined the Pythagorean fuzzy matrix (PFM) and its algebraic operations (Silambarasan and Sriram (2018)). They constructed nA and A^n of a Pythagorean fuzzy matrix A and investigated their algebraic properties. In (Silambarasan and Sriram (2019)), we have developed the Hamacher operations on Pythagorean fuzzy matrices and proved their algebraic properties.

This paper is organized as follows. In Section 2, we briefly introduced the Pythagorean fuzzy matrices and examples are given. In Section 3, we constructed Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations of Pythagorean fuzzy matrix A and investigated their algebraic properties. In Section 4, we prove some properties of necessity and possibility operators on Pythagorean fuzzy matrices. Section 5 concludes the paper with some future directions.

2. Pythagorean fuzzy matrices

In this section, we briefly introduce the Pythagorean fuzzy matrices and examples are given.

Definition 2.1.

A Pythagorean fuzzy matrix (PFM) is a matrix of pair $A = \left(\langle a_{ij}, a'_{ij}\rangle\right)$ of non negative real numbers $a_{ij}, a'_{ij} \in [0,1]$ satisfying the condition $0 \leq {a_{ij}}^2 + {a'_{ij}}^2 \leq 1$, for all i, j. Where $a_{ij} \in [0,1]$ is called the degree of membership and $a'_{ij} \in [0,1]$ is called the degree of non-membership.

Theorem 2.2.

The PFMs is larger than the set of IFMs.

Proof:

Any matrix $A = (a_{ij}, a'_{ij})$ that is an IFM is also a PFM. For any intuitionistic fuzzy matrices $A \in [0,1]$, we get $a_{ij}^2 \leq a_{ij}$ and ${a'_{ij}}^2 \leq a'_{ij}$. Thus $a_{ij} + a'_{ij} \leq 1 \Rightarrow a_{ij}^2 + {a'_{ij}}^2 \leq 1$. Consider a point $(0.8, 0.5)$, we see that $(0.8)^2 + (0.5)^2 \leq 1$. Thus, this is an PFM.

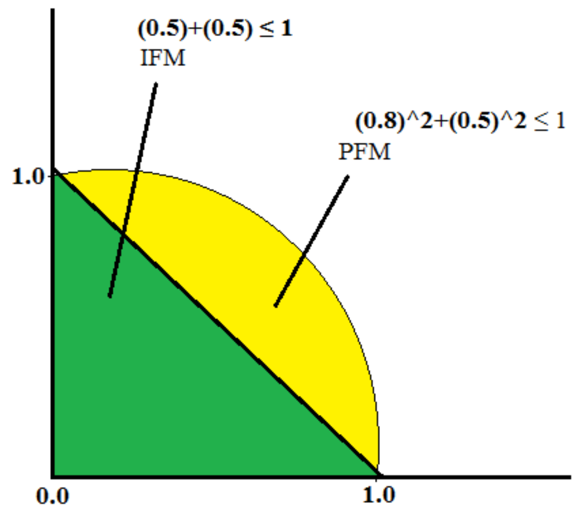

Figure 1. Comparison of space of IFMs and PFMs

For understanding the PFM better, we give an instance to illuminate the understandability of the PFM: We can definitely get $0.8 + 0.5 > 1$, and, therefore, it does not follow the condition of intuitionistic fuzzy matrices. Also, we can get $(0.8)^2 + (0.5)^2 = 0.64 + 0.25 = 0.89 \leq 1$, which is good enough to apply the PFM to control it. ■

Example 2.3.

$\mathbf{A} = \begin{bmatrix} (0.8, 0.5) & (0.2, 0.4) \\ (0.3, 0.4) & (0.4, 0.4) \end{bmatrix}$ is not an IFM, but it A is a PFM.

This development can be evidently recognized in Figure 1. Here we notice that IFMs are all points beneath the line $a_{ij} + a'_{ij} \leq 1$, the PFMs are all points with $a_{ij}^2 + {a'_{ij}}^2 \leq 1$. We see then that the PFMs enable for the presentation of a bigger body of nonstandard membership an IFMs.

3. Hamacher Scalar Multiplication and Hamacher Exponentiation Operations on Pythagorean Fuzzy Matrices

We defined the following operations over Hamacher operations on PFMs. In this section, we construct Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations on Pythagorean fuzzy matrix A and investigate their algebraic properties.

Definition 3.1.

Let $A = (\langle a_{ij}, a'_{ij} \rangle)$ and $B = (\langle b_{ij}, b'_{ij} \rangle)$ be any two Pythagorean fuzzy matrices of the same size, then we have:

(i) The Hamacher sum of A and B is defined by $A \oplus_H B = (c_{ij})$,

where

$$c_{ij} = \begin{cases} \langle 1, 0 \rangle, & if \ \langle a_{ij}, a'_{ij} \rangle = \langle 1, 0 \rangle, \langle b_{ij}, b'_{ij} \rangle = \langle 1, 0 \rangle \\ \left\langle \sqrt{\dfrac{a_{ij}^2 + b_{ij}^2 - 2a_{ij}^2 b_{ij}^2}{1 - a_{ij}^2 b_{ij}^2}}, \sqrt{\dfrac{{a'_{ij}}^2 {b'_{ij}}^2}{{a'_{ij}}^2 + {b'_{ij}}^2 - {a'_{ij}}^2 {b'_{ij}}^2}} \right\rangle, & \text{otherwise}, \end{cases}$$

for all i, j,

and

(ii) The Hamacher product of A and B is defined by $A \odot_H B = (c_{ij})$,

where

$$c_{ij} = \begin{cases} \langle 0, 1 \rangle, & if \ \langle a_{ij}, a'_{ij} \rangle = \langle 0, 1 \rangle, \langle b_{ij}, b'_{ij} \rangle = \langle 0, 1 \rangle \\ \left\langle \sqrt{\dfrac{a_{ij}^2 b_{ij}^2}{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}}, \sqrt{\dfrac{{a'_{ij}}^2 + {b'_{ij}}^2 - 2{a'_{ij}}^2 {b'_{ij}}^2}{1 - {a'_{ij}}^2 {b'_{ij}}^2}} \right\rangle, & \text{otherwise}, \end{cases}$$

for all i, j.

Based on the above definition, the Hamacher sum and Hamacher product over two PFMs A and B are further indicated as the following operations.

Theorem 3.2.

If n is any positive integer and A is a PFM, then the Hamacher scalar multiplication operation $(._h)$ is

$$n._h A = \underbrace{A \oplus_h \ldots \oplus_h A}_{n} = \left(\left\langle \sqrt{\frac{n a_{ij}^2}{1 + (n-1) a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n - (n-1) {a'_{ij}}^2}} \right\rangle \right). \tag{3.1}$$

Proof:

Mathematical induction can be used to prove that the above equation (3.1) holds for all positive integer n. The equation (3.1) is called ***P(n)***. Using the above definition of Hamacher sum (i), $A \oplus_H B$ we have:

$$A._h A = \left(\left\langle \sqrt{\frac{a_{ij}^2 + a_{ij}^2 - 2a_{ij}^2 a_{ij}^2}{1 - a_{ij}^2 a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2 {a'_{ij}}^2}{{a'_{ij}}^2 + {a'_{ij}}^2 - {a'_{ij}}^2 {a'_{ij}}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{2a_{ij}^2 - 2a_{ij}^4}{1-a_{ij}^4}}, \sqrt{\frac{{a'_{ij}}^4}{2{a'_{ij}}^2 - {a'_{ij}}^4}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{2a_{ij}^2(1 - a_{ij}^2)}{1-a_{ij}^4}}, \sqrt{\frac{{a'_{ij}}^4}{{a'_{ij}}^2(2 - {a'_{ij}}^2)}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{2a_{ij}^2(1 - a_{ij}^2)}{(1-a_{ij}^2)(1+a_{ij}^2)}}, \sqrt{\frac{{a'_{ij}}^2}{(2 - {a'_{ij}}^2)}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{2a_{ij}^2}{1+a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{(2 - {a'_{ij}}^2)}} \right\rangle\right),$$

$$2._hA = \left(\left\langle \sqrt{\frac{2a_{ij}^2}{1+(2-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{2-(2-1){a'_{ij}}^2}} \right\rangle\right), \quad \text{since } a_{ij}^2 = (2-1)a_{ij}^2,$$

$$n._hA = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle\right),$$

P(n) holds.

Suppose that Equation (3.1) holds for $n = m$,

$$\text{i.e.,} \quad m._hA = \underbrace{A \oplus_h ... \oplus_h A}_{m} = \left(\left\langle \sqrt{\frac{ma_{ij}^2}{1+(m-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{m-(m-1){a'_{ij}}^2}} \right\rangle\right).$$

Then,

$$(m+1)._hA = ((m._hA) \oplus_H A),$$

$$= \left[\sqrt{\frac{\frac{ma_{ij}^2}{1+(m-1)a_{ij}^2} + a_{ij}^2 - 2\frac{ma_{ij}^2}{1+(m-1)a_{ij}^2}.a_{ij}^2}{1 - \frac{ma_{ij}^2}{1+(m-1)a_{ij}^2}.a_{ij}^2}}, \sqrt{\frac{\frac{{a'_{ij}}^2}{m-(m-1){a'_{ij}}^2}.{a'_{ij}}^2}{\frac{{a'_{ij}}^2}{m-(m-1){a'_{ij}}^2} + {a'_{ij}}^2 - \frac{{a'_{ij}}^2}{m-(m-1){a'_{ij}}^2}.{a'_{ij}}^2}} \right],$$

$$= \left(\left\langle \sqrt{\frac{a_{ij}^2(m+1)(1-a_{ij}^2)}{(1+ma_{ij}^2)(1-a_{ij}^2)}}, \sqrt{\frac{({a'_{ij}}^2)^2}{{a'_{ij}}^2(m+1-m{a'_{ij}}^2)}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{(m+1)a_{ij}^2}{1+ma_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{m+1-m{a'_{ij}}^2}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{(m+1)a_{ij}^2}{1+[(m+1)-1]a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{m+1-[(m+1)-1]{a'_{ij}}^2}} \right\rangle\right).$$

So, when $n = m+1$,

$$n._h A = \underbrace{A \oplus_h \dots \oplus_h A}_{n} = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle\right),$$

also holds.

Using the induction hypothesis that *P(n)* holds for any positive integer n. ■

Theorem 3.3.

If n is any positive integer and A is a PFM, then the Hamacher exponentiation operation $(\wedge_h)$ is

$$A^{\wedge_h n} = \overbrace{A \odot_h \dots \odot_h A}^{n} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \right\rangle\right). \tag{3.2}$$

Proof:

Mathematical induction can be used to prove that the above Equation (3.2) holds for all positive integer n. The equation (3.2) is called *P(n)*. Using the above definition of Hamacher product (ii), $A \odot_H B$ we have:

$$A^{\wedge_h A} = \left(\left\langle \sqrt{\frac{a_{ij}^2 a_{ij}^2}{a_{ij}^2 + a_{ij}^2 - a_{ij}^2 a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2 + {a'_{ij}}^2 - 2{a'_{ij}}^2 {a'_{ij}}^2}{1-{a'_{ij}}^2 {a'_{ij}}^2}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{{a_{ij}}^4}{2{a_{ij}}^2 - {a_{ij}}^4}}, \sqrt{\frac{2{a'_{ij}}^2 - 2{a'_{ij}}^4}{1-{a'_{ij}}^4}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{{a_{ij}}^4}{{a_{ij}}^2(2-{a_{ij}}^2)}}, \sqrt{\frac{2{a'_{ij}}^2(1-{a'_{ij}}^2)}{1-{a'_{ij}}^4}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{{a_{ij}}^2}{(2-{a_{ij}}^2)}}, \sqrt{\frac{2{a'_{ij}}^2(1-{a'_{ij}}^2)}{(1-{a'_{ij}}^2)(1+{a'_{ij}}^2)}} \right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{{a_{ij}}^2}{(2-{a_{ij}}^2)}}, \sqrt{\frac{2{a'_{ij}}^2}{1+{a'_{ij}}^2}} \right\rangle\right),$$

$$A^{\wedge_h 2} = \left(\left\langle \sqrt{\frac{{a_{ij}}^2}{2-(2-1){a_{ij}}^2}}, \sqrt{\frac{2{a'_{ij}}^2}{1+(2-1){a'_{ij}}^2}} \right\rangle\right), \text{ since } a_{ij}^2 = (2-1)a_{ij}^2,$$

$$A^{\wedge_h n} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \right\rangle\right),$$

P(n) holds.

Suppose that Equation (3.2) holds for $n = m$,

$$\text{i.e., } A^{\wedge_h m} = \overbrace{A \odot_h \ldots \odot_h A}^{m} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{m-(m-1)a_{ij}^2}}, \sqrt{\frac{m{a'_{ij}}^2}{1+(m-1){a'_{ij}}^2}} \right\rangle\right).$$

So, when $n = m+1$,

$$A^{\wedge_h m+1} = \left(\left\langle \sqrt{\frac{{a_{ij}}^2}{m+1-[(m+1)-1]{a_{ij}}^2}}, \sqrt{\frac{(m+1){a'_{ij}}^2}{1+[(m+1)-1]{a'_{ij}}^2}} \right\rangle\right),$$

$$A^{\wedge_h n} = \overbrace{A \odot_h \ldots \odot_h A}^{n} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \right\rangle\right),$$

also holds.

Using the induction hypothesis that *P(n)* holds for any positive integer n. ■

Next, we prove the result of $(n._h A)$ and $(A^{\wedge_h n})$ are also PFMs.

Theorem 3.4.

For any PFM A and for any positive integer n, then $(n._h A)$ and $(A^{\wedge_h n})$ are PFMs.

Proof:

Since $0 \le a_{ij}^2 \le 1$, $0 \le {a'_{ij}}^2 \le 1$, $0 \le a_{ij}^2 + {a'_{ij}}^2 \le 1$, and $n > 1$, we have:

$$(n-1)a_{ij}^2 > -1, \quad 1+(n-1)a_{ij}^2 > 0,$$

$$n-(n-1){a'_{ij}}^2 = (1-{a'_{ij}}^2)n + {a'_{ij}}^2 > {a'_{ij}}^2 \ge 0.$$

Then, it is easy to get that $\sqrt{\dfrac{na_{ij}^2}{1+(n-1)a_{ij}^2}} \ge 0$, $\sqrt{\dfrac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \ge 0$.

Considering that $1+(n-1)a_{ij}^2 = na_{ij}^2 + 1 - a_{ij}^2 \ge na_{ij}^2$ and

$$n-(n-1){a'_{ij}}^2 = {a'_{ij}}^2 + n(1-{a'_{ij}}^2) \geq {a'_{ij}}^2, \text{ we get:}$$

$$\sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} \leq 1, \quad \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \leq 1.$$

For $a_{ij}^2 + {a'_{ij}}^2 \leq 1, \quad 0 \leq {a'_{ij}}^2 \leq 1 - a_{ij}^2$, it can be found that:

$$\Rightarrow \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} + \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}}$$

$$= \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} + \sqrt{\frac{1}{\frac{n}{{a'_{ij}}^2}-(n-1)}}$$

$$\leq \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} + \sqrt{\frac{1}{\frac{n}{1-{a_{ij}^2}^2}-(n-1)}} = 1.$$

Thus,

$$0 \leq \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} \leq 1, \quad 0 \leq \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \leq 1,$$

$$\Rightarrow \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} + \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \leq 1.$$

Similarly, we can also get

$$0 \leq \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}} \leq 1, \quad 0 \leq \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \leq 1,$$

$$\Rightarrow \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}} + \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \leq 1.$$

Hence, $(n._h)A$ and $(A^{\wedge_h n})$ are PFMs. ■

Theorem 3.5.

Let $A = \left(\langle a_{ij}, a'_{ij}\rangle\right), B = \left(\langle b_{ij}, b'_{ij}\rangle\right)$ be any two PFMs of the same size and for any positive integers n, n_1, n_2.

(i) $n_1._hA \oplus_H n_2._hA = (n_1+n_2)._hA$,

(ii) $(n._hA) \oplus_H (n._hB) = n._h(A \oplus_H B)$,

$(iii)\quad A^{\wedge_h n_1} \odot_H A^{\wedge_h n_2} = A^{\wedge_h (n_1+n_2)}$,

$(iv)\quad A^{\wedge_h n} \odot_H B^{\wedge_h n} = (A \odot_H B)^{\wedge_h n}$,

$(v)\quad n_2 \cdot_h (n_1 \cdot_h A) = (n_1 n_2) \cdot_h A$,

$(vi)\quad (A^{\wedge_h n_1})^{\wedge_h n_2} = A^{\wedge_h (n_1 n_2)}$.

Proof:

In the following, we shall prove $(i), (ii), (v)$ and $(iii), (iv), (vi)$ can be proved analogously.

(i) By Equations (3.1) and (3.2), we have:

$$n_1 \cdot_h A = \left(\left\langle \sqrt{\frac{n_1 a_{ij}^2}{1+(n_1-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n_1-(n_1-1){a'_{ij}}^2}} \right\rangle\right) = (\langle b_{ij}, b'_{ij}\rangle),$$

$$n_2 \cdot_h A = \left(\left\langle \sqrt{\frac{n_2 a_{ij}^2}{1+(n_2-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n_2-(n_2-1){a'_{ij}}^2}} \right\rangle\right) = (\langle c_{ij}, c'_{ij}\rangle),$$

$$B \oplus_H C = (n_1 \cdot_h A) \oplus_H (n_2 \cdot_h A),$$

$$B \oplus_H C = \left(\left\langle \sqrt{\frac{b_{ij}^2 + c_{ij}^2 - 2b_{ij}^2 c_{ij}^2}{1-b_{ij}^2 c_{ij}^2}}, \sqrt{\frac{{b'_{ij}}^2 {c'_{ij}}^2}{{b'_{ij}}^2 + {c'_{ij}}^2 - {b'_{ij}}^2 {c'_{ij}}^2}} \right\rangle\right).$$

We can further get:

$$\sqrt{\frac{b_{ij}^2 + c_{ij}^2 - 2b_{ij}^2 c_{ij}^2}{1-b_{ij}^2 c_{ij}^2}}$$

$$= \sqrt{\frac{\dfrac{n_1 a_{ij}^2}{1+(n_1-1)a_{ij}^2} + \dfrac{n_2 a_{ij}^2}{1+(n_2-1)a_{ij}^2} - 2\dfrac{n_1 a_{ij}^2}{1+(n_1-1)a_{ij}^2}\dfrac{n_2 a_{ij}^2}{1+(n_2-1)a_{ij}^2}}{1 - \dfrac{n_1 a_{ij}^2}{1+(n_1-1)a_{ij}^2}\dfrac{n_2 a_{ij}^2}{1+(n_2-1)a_{ij}^2}}}$$

$$= \sqrt{\frac{n_1 a_{ij}^2(1+n_2 a_{ij}^2 - a_{ij}^2) + n_2 a_{ij}^2(1+n_1 a_{ij}^2 - a_{ij}^2) - 2n_1 a_{ij}^2 n_2 a_{ij}^2}{(1+n_1 a_{ij}^2 - a_{ij}^2)(1+n_2 a_{ij}^2 - a_{ij}^2) - n_1 a_{ij}^2 n_2 a_{ij}^2}}$$

$$= \sqrt{\frac{(n_1+n_2)a_{ij}^2 - (n_1+n_2)a_{ij}^4}{1+(n_1+n_2-2)a_{ij}^2 - (n_1+n_2-1)a_{ij}^4}}$$

$$= \sqrt{\frac{(n_1+n_2)a_{ij}^2(1-a_{ij}^2)}{(1+(n_1+n_2-1)a_{ij}^2)(1-a_{ij}^2)}}$$

$$n_1 \cdot_h A = \sqrt{\frac{(n_1+n_2)a_{ij}^2}{1+(n_1+n_2-1)a_{ij}^2}},$$

and

$$\sqrt{\frac{{b'_{ij}}^2{c'_{ij}}^2}{{b'_{ij}}^2+{c'_{ij}}^2-{b'_{ij}}^2{c'_{ij}}^2}}$$

$$= \sqrt{\frac{\frac{{a'_{ij}}^2}{n_1-(n_1-1){a'_{ij}}^2}\frac{{a'_{ij}}^2}{n_2-(n_2-1){a'_{ij}}^2}}{\frac{{a'_{ij}}^2}{n_1-(n_1-1){a'_{ij}}^2}+\frac{{a'_{ij}}^2}{n_2-(n_2-1){a'_{ij}}^2}-\frac{{a'_{ij}}^2}{n_1-(n_1-1){a'_{ij}}^2}\frac{{a'_{ij}}^2}{n_2-(n_2-1){a'_{ij}}^2}}}$$

$$= \sqrt{\frac{{a'_{ij}}^2{a'_{ij}}^2}{{a'_{ij}}^2(n_2+{a'_{ij}}^2-n_2{a'_{ij}}^2)+{a'_{ij}}^2(n_1+{a'_{ij}}^2-n_1{a'_{ij}}^2)-{a'_{ij}}^2{a'_{ij}}^2}}$$

$$= \sqrt{\frac{{a'_{ij}}^2}{(n_2+{a'_{ij}}^2-n_2{a'_{ij}}^2)+(n_1+{a'_{ij}}^2-n_1{a'_{ij}}^2)-{a'_{ij}}^2}}$$

$$n_2 \cdot_h A = \sqrt{\frac{{a'_{ij}}^2}{(n_1+n_2)-(n_1+n_2-1){a'_{ij}}^2}}.$$

Since $(n_1+n_2)._h A = \left(\left\langle \sqrt{\frac{(n_1+n_2)a_{ij}^2}{1+(n_1+n_2-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{(n_1+n_2)-(n_1+n_2-1){a'_{ij}}^2}} \right\rangle\right),$

we can finally get $(n_1 \cdot_h A) \oplus_H (n_2 \cdot_h A) = (n_1+n_2)._h A.$

(ii) By Equations (3.1) and (3.2), we have:

$$n._h A = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle\right) = (\langle b_{ij}, b'_{ij}\rangle),$$

$$n._h B = \left(\left\langle \sqrt{\frac{nb_{ij}^2}{1+(n-1)b_{ij}^2}}, \sqrt{\frac{{b'_{ij}}^2}{n-(n-1){b'_{ij}}^2}} \right\rangle\right) = (\langle c_{ij}, c'_{ij}\rangle),$$

$$B \oplus_H C = (n._h A) \oplus_H (n._h B),$$

$$B \oplus_H C = \left(\left\langle \sqrt{\frac{b_{ij}^2+c_{ij}^2-2b_{ij}^2c_{ij}^2}{1-b_{ij}^2c_{ij}^2}}, \sqrt{\frac{{b'_{ij}}^2{c'_{ij}}^2}{{b'_{ij}}^2+{c'_{ij}}^2-{b'_{ij}}^2{c'_{ij}}^2}} \right\rangle\right).$$

We can further get:

$$\sqrt{\frac{b_{ij}^2+c_{ij}^2-2b_{ij}^2c_{ij}^2}{1-b_{ij}^2c_{ij}^2}}$$

$$=\sqrt{\frac{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}+\frac{nb_{ij}^2}{1+(n-1)b_{ij}^2}-2\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}\frac{nb_{ij}^2}{1+(n-1)b_{ij}^2}}{1-\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}\frac{nb_{ij}^2}{1+(n-1)b_{ij}^2}}}$$

$$=\sqrt{\frac{na_{ij}^2(1+nb_{ij}^2-b_{ij}^2)+nb_{ij}^2(1+na_{ij}^2-a_{ij}^2)-2na_{ij}^2nb_{ij}^2}{(1+nb_{ij}^2-b_{ij}^2)(1+na_{ij}^2-a_{ij}^2)-na_{ij}^2nb_{ij}^2}}$$

$$(n._hA)=\sqrt{\frac{na_{ij}^2+nb_{ij}^2-2na_{ij}^2b_{ij}^2}{1+(n-1)(a_{ij}^2+b_{ij}^2)-(2n-1)a_{ij}^2b_{ij}^2}},$$

and

$$\sqrt{\frac{{b'_{ij}}^2{c'_{ij}}^2}{{b'_{ij}}^2+{c'_{ij}}^2-{b'_{ij}}^2{c'_{ij}}^2}}$$

$$=\sqrt{\frac{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}\frac{{b'_{ij}}^2}{n-(n-1){b'_{ij}}^2}}{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}+\frac{{b'_{ij}}^2}{n-(n-1){b'_{ij}}^2}-\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}\frac{{b'_{ij}}^2}{n-(n-1){b'_{ij}}^2}}}$$

$$=\sqrt{\frac{{a'_{ij}}^2{b'_{ij}}^2}{{a'_{ij}}^2(n+{b'_{ij}}^2-n{b'_{ij}}^2)+{b'_{ij}}^2(n+{a'_{ij}}^2-n{a'_{ij}}^2)-{a'_{ij}}^2{b'_{ij}}^2}}$$

$$(n._hB)=\sqrt{\frac{{a'_{ij}}^2{b'_{ij}}^2}{n({a'_{ij}}^2+{b'_{ij}}^2)-(2n-1){a'_{ij}}^2{b'_{ij}}^2}}.$$

Thus,

$$n._h(A\oplus_H B)$$

$$=\left(\left\langle\sqrt{\frac{n\frac{a_{ij}^2+b_{ij}^2-2a_{ij}^2b_{ij}^2}{1-a_{ij}^2b_{ij}^2}}{1+(n-1)\frac{a_{ij}^2+b_{ij}^2-2a_{ij}^2b_{ij}^2}{1-a_{ij}^2b_{ij}^2}}},\sqrt{\frac{\frac{{a'_{ij}}^2{b'_{ij}}^2}{{a'_{ij}}^2+{b'_{ij}}^2-{a'_{ij}}^2{b'_{ij}}^2}}{n-(n-1)\frac{{a'_{ij}}^2{b'_{ij}}^2}{{a'_{ij}}^2+{b'_{ij}}^2-{a'_{ij}}^2{b'_{ij}}^2}}}\right\rangle\right),$$

$$= \left(\left\langle \sqrt{\frac{na_{ij}^2 + nb_{ij}^2 - 2na_{ij}^2 b_{ij}^2}{1 + (n-1)(a_{ij}^2 + b_{ij}^2) - (2n-1)a_{ij}^2 b_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2 {b'_{ij}}^2}{n({a'_{ij}}^2 + {b'_{ij}}^2) - (2n-1){a'_{ij}}^2 {b'_{ij}}^2}} \right\rangle \right).$$

Comparing above results, we can finally get

$$(n\cdot_h A) \oplus_H (n\cdot_h B) = n\cdot_h (A \oplus_H B).$$

(v) By Equations (3.1) and (3.2), we have:

$$n_1\cdot_h A = \left(\left\langle \sqrt{\frac{n_1 a_{ij}^2}{1 + (n_1 - 1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n_1 - (n_1 - 1){a'_{ij}}^2}} \right\rangle \right) = \left(\langle b_{ij}, b'_{ij} \rangle \right),$$

$$n_2\cdot_h (n_1\cdot_h A) = \left(\left\langle \sqrt{\frac{n_2 b_{ij}}{1 + (n_2 - 1)b_{ij}}}, \sqrt{\frac{b'_{ij}}{n_2 - (n_2 - 1)b'_{ij}}} \right\rangle \right).$$

We can further get:

$$\sqrt{\frac{n_2 b_{ij}}{1 + (n_2 - 1)b_{ij}}}$$

$$= \sqrt{\frac{n_2 \dfrac{n_1 a_{ij}^2}{1 + (n_1 - 1)a_{ij}^2}}{1 + (n_2 - 1)\dfrac{n_1 a_{ij}^2}{1 + (n_1 - 1)a_{ij}^2}}} = \sqrt{\frac{n_1 n_2 a_{ij}^2}{1 + (n_1 n_2 - 1)a_{ij}^2}},$$

and

$$\sqrt{\frac{b'_{ij}}{n_2 - (n_2 - 1)b'_{ij}}}$$

$$= \sqrt{\frac{\dfrac{{a'_{ij}}^2}{n_1 - (n_1 - 1){a'_{ij}}^2}}{n_2 - (n_2 - 1)\dfrac{{a'_{ij}}^2}{n_1 - (n_1 - 1){a'_{ij}}^2}}} = \sqrt{\frac{{a'_{ij}}^2}{n_1 n_2 - (n_1 n_2 - 1){a'_{ij}}^2}}.$$

Since $n_1(n_2\cdot_h)A = \left(\left\langle \sqrt{\frac{n_1 n_2 a_{ij}^2}{1 + (n_1 n_2 - 1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n_1 n_2 - (n_1 n_2 - 1){a'_{ij}}^2}} \right\rangle \right),$

we can finally get $n_2\cdot_h (n_1\cdot_h A) = (n_1 n_2)\cdot_h A.$ ■

Theorem 3.6.

Let $A = (\langle a_{ij}, a'_{ij}\rangle)$, $B = (\langle b_{ij}, b'_{ij}\rangle)$ be any two PFMs of the same size and for any positive integer n.

(i) $n\cdot_h(A \wedge B) = (n\cdot_h A) \wedge (n\cdot_h B)$,

(ii) $n\cdot_h(A \vee B) = (n\cdot_h A) \vee (n\cdot_h B)$,

(iii) $(A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}$,

(iv) $(A \vee B)^{\wedge_h n} = A^{\wedge_h n} \vee B^{\wedge_h n}$.

Proof:

In the following, we shall prove $(ii), (iv)$ and $(i), (iii)$ can be proved analogously.

(i) Since $(A \wedge B) = (\langle \min\{a_{ij}, b_{ij}\}, \max\{a'_{ij}, b'_{ij}\}\rangle)$,

$$n\cdot_h(A \wedge B) = (\langle c_{ij}, c'_{ij}\rangle), n\cdot_h A = (\langle d_{ij}, d'_{ij}\rangle), n\cdot_h B = (\langle e_{ij}, e'_{ij}\rangle),$$

where

$$c_{ij} = \sqrt{\frac{n(\min\{a_{ij}^2, b_{ij}^2\})}{1 + (n-1)(\min\{a_{ij}^2, b_{ij}^2\})}}, \quad \text{and} \quad c'_{ij} = \sqrt{\frac{(\max\{{a'_{ij}}^2, {b'_{ij}}^2\})}{n - (n-1)(\max\{{a'_{ij}}^2, {b'_{ij}}^2\})}},$$

we have

$$\begin{aligned} c_{ij} &= \sqrt{\frac{n(\min\{a_{ij}^2, b_{ij}^2\})}{1 + (n-1)(\min\{a_{ij}^2, b_{ij}^2\})}} \\ &= \min\left\{\sqrt{\frac{na_{ij}^2}{1 + (n-1)a_{ij}^2}}, \sqrt{\frac{nb_{ij}^2}{1 + (n-1)b_{ij}^2}}\right\}, \\ &= \min\{d_{ij}, e_{ij}\}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} c'_{ij} &= \sqrt{\frac{(\max\{{a'_{ij}}^2, {b'_{ij}}^2\})}{n - (n-1)(\max\{{a'_{ij}}^2, {b'_{ij}}^2\})}} \\ &= \max\left\{\sqrt{\frac{{a'_{ij}}^2}{n - (n-1){a'_{ij}}^2}}, \sqrt{\frac{{b'_{ij}}^2}{n - (n-1){b'_{ij}}^2}}\right\}, \\ &= \max\{d'_{ij}, e'_{ij}\}. \end{aligned} \tag{3.4}$$

Comparing Equations (3.3) and (3.4), we get:

$$(n._h A) \wedge (n._h B) = \left(\left\langle \min\left\{d_{ij}, d'_{ij}\right\}, \max\left\{e_{ij}, e'_{ij}\right\}\right\rangle\right),$$

$$= \left[\min\left\{\sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}}\right\}, \right.$$

$$\left. \max\left\{\sqrt{\frac{nb_{ij}^2}{1+(n-1)b_{ij}^2}}, \sqrt{\frac{{b'_{ij}}^2}{n-(n-1){b'_{ij}}^2}}\right\} \right].$$

Thus, we have $n._h(A \wedge B) = (n._h A) \wedge (n._h B)$.

(iii) Since $(A \wedge B) = \left(\left\langle \min\left\{a_{ij}, b_{ij}\right\}, \max\left\{a'_{ij}, b'_{ij}\right\}\right\rangle\right)$, then

$$(A \wedge B)^{\wedge_h n} = \left(\left\langle c_{ij}, c'_{ij}\right\rangle\right), A^{\wedge_h n} = \left(\left\langle d_{ij}, d'_{ij}\right\rangle\right), B^{\wedge_h n} = \left(\left\langle e_{ij}, e'_{ij}\right\rangle\right),$$

where

$$c_{ij} = \sqrt{\frac{(\min\left\{a_{ij}^2, b_{ij}^2\right\})}{n-(n-1)(\min\left\{a_{ij}^2, b_{ij}^2\right\})}}, \text{ and } c'_{ij} = \sqrt{\frac{n(\max\left\{{a'_{ij}}^2, {b'_{ij}}^2\right\})}{1+(n-1)(\max\left\{{a'_{ij}}^2, {b'_{ij}}^2\right\})}},$$

we have

$$c_{ij} = \sqrt{\frac{(\min\left\{a_{ij}^2, b_{ij}^2\right\})}{n-(n-1)(\min\left\{a_{ij}^2, b_{ij}^2\right\})}}$$

$$= \min\left\{\sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{b_{ij}^2}{n-(n-1)b_{ij}^2}}\right\}$$

$$= \min\left\{d_{ij}, e_{ij}\right\}, \tag{3.5}$$

and

$$c'_{ij} = \sqrt{\frac{n(\max\left\{{a'_{ij}}^2, {b'_{ij}}^2\right\})}{1+(n-1)(\max\left\{{a'_{ij}}^2, {b'_{ij}}^2\right\})}}$$

$$= \max\left\{\sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}}, \sqrt{\frac{n{b'_{ij}}^2}{1+(n-1){b'_{ij}}^2}}\right\}$$

$$= \max\left\{d'_{ij}, e'_{ij}\right\}. \tag{3.6}$$

Comparing Equations (3.5) and (3.6), we get:

$$A^{\wedge_h n} \wedge B^{\wedge_h n} = \left(\left\langle \min\left\{d_{ij}, d'_{ij}\right\}, \max\left\{e_{ij}, e'_{ij}\right\}\right\rangle\right),$$

$$= \left[\min\left\{ \sqrt{\frac{{a_{ij}}^2}{n-(n-1){a_{ij}}^2}}, \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \right\}, \right.$$

$$\left. \max\left\{ \sqrt{\frac{{b_{ij}}^2}{n-(n-1){b_{ij}}^2}}, \sqrt{\frac{n{b'_{ij}}^2}{1+(n-1){b'_{ij}}^2}} \right\} \right].$$

Hence, $(A \wedge B)^{\wedge_h n} = A^{\wedge_h n} \wedge B^{\wedge_h n}$. ∎

4. Necessity and Possibility operators on Pythagorean fuzzy matrices

In this section, some properties of necessity and possibility operators on Pythagorean fuzzy matrices are verify.

Definition 4.1.

For any Pythagorean fuzzy matrix A, the necessity $(\Box)$ and the possibility $(\Diamond)$ operators are defined as follows:

$$\Box A = \left(\left\langle a_{ij}, \sqrt{1-a_{ij}^2} \right\rangle\right),$$

$$\Diamond A = \left(\left\langle \sqrt{1-{a'_{ij}}^2}, a'_{ij} \right\rangle\right).$$

Theorem 4.2.

For any PFM A and for any positive integer n,

(i) $\Box(n._h A) = n._h(\Box A)$,

(ii) $\Diamond(n._h A) = n._h(\Diamond A)$,

(iii) $\Box A^{\wedge_h n} = (\Box A)^{\wedge_h n}$,

(iv) $\Diamond A^{\wedge_h n} = (\Diamond A)^{\wedge_h n}$.

Proof:

$$(i)\ \Box(n._h A) = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{1-\left(\sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}\right)^2} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{1-\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{1+(n-1)a_{ij}^2-na_{ij}^2}{1+(n-1)a_{ij}^2}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{1+na_{ij}^2 - a_{ij}^2 - na_{ij}^2}{1+(n-1)a_{ij}^2}} \right\rangle \right)$$

$$\Box(n._h A) = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{1-a_{ij}^2}{1+(n-1)a_{ij}^2}} \right\rangle \right),$$

$$n._h(\Box A) = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{\left(\sqrt{1-a_{ij}^2}\right)^2}{n-(n-1)\left(\sqrt{1-a_{ij}^2}\right)^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{1-a_{ij}^2}{n-(n-1)\left(1-a_{ij}^2\right)}} \right\rangle \right),$$

$$n._h(\Box A) = \left(\left\langle \sqrt{\frac{na_{ij}^2}{1+(n-1)a_{ij}^2}}, \sqrt{\frac{1-a_{ij}^2}{1+(n-1)a_{ij}^2}} \right\rangle \right).$$

Hence, $\Box(n._h A) = n._h(\Box A)$.

$$(ii)\ \Diamond(n._h A) = \left(\left\langle \sqrt{1-\left(\sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}}\right)^2}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{1-\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{n-(n-1){a'_{ij}}^2 - {a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{n-n{a'_{ij}}^2 + {a'_{ij}}^2 - {a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right),$$

$$\Diamond(n._h A) = \left(\left\langle \sqrt{\frac{n(1-{a'_{ij}}^2)}{n-(n-1){a'_{ij}}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right),$$

$$n._h(\Diamond A) = \left(\left\langle \sqrt{\frac{n(\sqrt{1-{a'_{ij}}^2})^2}{1+(n-1)(\sqrt{1-{a'_{ij}}^2})^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{n(1-{a'_{ij}}^2)}{1+(n-1)(1-{a'_{ij}}^2)}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right),$$

$$n._h(\Diamond A) = \left(\left\langle \sqrt{\frac{n(1-{a'_{ij}}^2)}{n-(n-1){a'_{ij}}^2}}, \sqrt{\frac{{a'_{ij}}^2}{n-(n-1){a'_{ij}}^2}} \right\rangle \right).$$

Hence, $\Diamond(n._h A) = n._h(\Diamond A)$.

$$(iii)\ \ \Box A^{\wedge_h n} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{1-\left(\sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}\right)^2} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{1-\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n-(n-1)a_{ij}^2-a_{ij}^2}{n-(n-1)a_{ij}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n-na_{ij}^2+a_{ij}^2-a_{ij}^2}{n-(n-1)a_{ij}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n(1-a_{ij}^2)}{n-(n-1)a_{ij}^2}} \right\rangle \right),$$

$$(\Box A)^{\wedge_h n} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n(\sqrt{1-a_{ij}^2})^2}{n-(n-1)(\sqrt{1-a_{ij}^2})^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n(1-a_{ij}^2)}{n-(n-1)(1-a_{ij}^2)}} \right\rangle \right),$$

$$(\Box A)^{\wedge_h n} = \left(\left\langle \sqrt{\frac{a_{ij}^2}{n-(n-1)a_{ij}^2}}, \sqrt{\frac{n(1-a_{ij}^2)}{n-(n-1)a_{ij}^2}} \right\rangle \right).$$

Hence, $\Box A^{\wedge_h n} = (\Box A)^{\wedge_h n}$.

$$(iv)\ \ \Diamond A^{\wedge_h n} = \left(\left\langle \sqrt{1-\left(\sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}}\right)^2}, \sqrt{\frac{n{a'_{ij}}^2}{1+(n-1){a'_{ij}}^2}} \right\rangle \right)$$

$$= \left(\left\langle \sqrt{1 - \frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}}, \sqrt{\frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{1+(n-1)a'^{2}_{ij} - na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}}, \sqrt{\frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}} \right\rangle\right),$$

$$\Diamond A^{\wedge_h n} = \left(\left\langle \sqrt{\frac{1-a'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}}, \sqrt{\frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}} \right\rangle\right),$$

$$(\Diamond A)^{\wedge_h n} = \left(\left\langle \sqrt{\frac{\left(\sqrt{1-a'^{2}_{ij}}\right)^2}{1+(n-1)\left(\sqrt{1-a'^{2}_{ij}}\right)^2}}, \sqrt{\frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}} \right\rangle\right)$$

$$= \left(\left\langle \sqrt{\frac{\left(1-a'^{2}_{ij}\right)}{1+(n-1)\left(1-a'^{2}_{ij}\right)}}, \sqrt{\frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}} \right\rangle\right),$$

$$(\Diamond A)^{\wedge_h n} = \left(\left\langle \sqrt{\frac{1-a'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}}, \sqrt{\frac{na'^{2}_{ij}}{1+(n-1)a'^{2}_{ij}}} \right\rangle\right).$$

Hence, $\Diamond A^{\wedge_h n} = (\Diamond A)^{\wedge_h n}$, which completes the proof of this theorem. ■

5. Conclusion

The work has extended the Hamacher operation results under Pythagorean fuzzy environment. We briefly introduced the Pythagorean fuzzy matrices and some theorems and examples are applied to illustrate the performance of the proposed methods. Then we constructed Hamacher scalar multiplication $(n._h A)$ and Hamacher exponentiation $(A^{\wedge_h n})$ operations on Pythagorean fuzzy matrix A and investigated their algebraic properties. Further, some properties of necessity and possibility operators on Pythagorean fuzzy matrices are verified.

It is worthwhile to point out that the proposed Hamacher operations over PFMs will be applied to aggregating Pythagorean fuzzy information in the future.

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REFERENCES

Emam, E.G. and Fndh, M.A. (2016). Some results associated wiith the max-min and min-max compositions of bifuzzy matrices, Journal of the Egyption Mathematical Society, Vol. 24, No. 4, pp. 515-521.

Im, Y.P., Lee, F.B. and Park, S.W. (2001). The determinant of square intuitionistic fuzzy matrices, Far East Journal of Mathematical Science, Vol. 3, No. 5, pp. 789-796.

Khan, S.K. and Pal, M. (2006). Some operations on Intuitionistic Fuzzy Matrices, Acta Ciencia Indica, Vol. 32, pp. 515-524.

Khan, S.K, Pal, M. and Shyamal, A.K. (2002). Intuitionistic Fuzzy Matrices, Notes on Intuitionistic Fuzzy Sets, Vol. 8, No. 2, pp. 51-62.

Mondal, S. and Pal, M. (2013). Similarity relations, invertibility and eigenvalues of IFM, Fuzzy Information and Engineering, Vol. 5, No. 4, pp. 431-443.

Muthuraji, T. and Sriram, S. (2017). Representation and Decomposition of an Intuitionistic fuzzy matrix using some (α, α') cuts, Applications and Applied Mathematics , Vol. 12, No. 1, pp. 241-258.

Muthuraji, T., Sriram, S. and Murugadas, P. (2016). Decomposition of intuitionistic fuzzy matrices, Fuzzy Information and Engineering, Vol. 8, No. 3, pp. 345-354.

Pal, P. (2001). Intuitionistic fuzzy determinant, V.U.J. Physical Sciences, Vol. 7, pp. 87-93.

Peng, X and Yang, Y. (2015). Some results for Pythagorean fuzzy sets, Int J Intell Syst, Vol. 30, No. 11, pp. 1133-1160.

Silambarasan, I. and Sriram, S. (2018). Algebraic operations on Pythagorean fuzzy matrices, Mathematical Sciences International Research Journal, Vol. 7, No. 2, pp. 406-418.

Silambarasan, I. and Sriram, S. (2019). Hamacher operations on Pythagorean Fuzzy Matrices, Journal of Applied Mathematics and Computational Mechanics, Vol. 18, No. 3, pp. 69-78.

Sriram, S. and Boobalan, J. (2016). Monoids of intuitionistic fuzzy matrices, Annals of Fuzzy Mathematics and Informatics, Vol. 11, No. 3, pp. 505-510.

Thomason, M.G. (1977). Convergence of powers of Fuzzy matrix, J. Mathematical Analysis and Applications, Vol. 57, No. 2, pp. 476-480.

Yager, R.R. (2013). Pythagorean fuzzy subsets. In *Proc Joint IFSA World Congress and NAFIPS Annual Meeting*, Edmonton, Canada, pp. 57-61.

Yager, R.R. (2014). Pythagorean membership grades in multi-criteria decision making, IEEE Transactions on Fuzzy Systems, Vol. 22, No. 4, pp. 958-965.

Zhang, X. and Xu, Z. (2012). A new method for ranking intuitionistic fuzzy values and its application in multi attribute decision making, Fuzzy Optimization Decision Making, Vol. 11, No. 2, pp. 135-146.

Zhang, X.L. and Xu, Z.S. (2014). Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, Int. J. of Intelligent Systems, Vol. 29, No. 12, pp. 1061-1078.