Applications and Applied Mathematics: An International Journal (AAM)

Volume 15 | Issue 1 | Article 13

6-2020

# On Fejér type inequalities for convex mappings utilizing generalized fractional integrals

A. Kashuri
*University "Ismail Qermali"*

R. Liko
*University "Ismail Qermali"*

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 15, Issue 1 (June 2020), pp. 240 - 255

# On Fejér type inequalities for convex mappings utilizing generalized fractional integrals

**[1]A. Kashuri and [2]R. Liko**

Department of Mathematics
Faculty of Technical Science
University "Ismail Qermali"
Vlora, Albania
[1]artionkashuri@gmail.com; [2]rozanaliko86@gmail.com

Received: November 25, 2019; Accepted: April 27, 2020

## Abstract

In this work, we first establish Hermite-Hadamard-Fejér type inequalities for convex function involving generalized fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Moreover, we obtain some trapezoid type inequalities for these kind of generalized fractional integrals. The results given in this paper provide generalization of several inequalities obtained in earlier studies.

**Keywords:** Hermite-Hadamard-Fejér inequality; generalized fractional integrals; convex functions

**MSC 2010 No.:** 26A51, 26A33, 26D07, 26D10, 26D15

## 1. Introduction

The following inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied, see (Azpeitia (1994); Bakula and Pečarič (2004); Budak and Sarikaya (2016)); (Chu et al. (2016))-( Dragomir and Agarwal (1998)); (Iqbal et al. (2018)); (Jleli and Samet (2016))-( Khan et al. (2019)); (Khurshid et al. (2018))-(Rassias and Kashuri (2019)); (Sarikaya and Budak (2016))-(Sarikaya and Ertuğral

(2019)); (Sarikaya et al. (2013); Set et al. (2014); Set et al. (2016)); (Wang et al. (2013))-(Zhang et al. (2015)).

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities, see (Fejér (1906)). For some recent results connected with Hermite-Hadamard-Fejér type inequalities, see (Ali et al. (2017); Bombardelli and Varosaneč (2009); Chen and Katugampola (2017); Chen and Wu (2014); Fejér (1906); Işcan (2015); Khurshid et al. (2019); Sarikaya (2012); Sarikaya and Budak (2017); Sarikaya et al. (2014); Set et al. (2015); Tseng et al. (2011)).

## 2. Preliminaries

In the following we recall some useful known definitions and results:

Let $f \in L[a,b]$. The Riemann-Liouville fractional integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt\,, \quad x > a \tag{1}$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt\,, \quad x < b, \tag{2}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In (Sarikaya and Budak (2014)), Sarikaya et al. first proved the following important Hermite-Hadamard type utilizing Riemann-Liouville fractional integrals: if $f: [a,b] \longrightarrow \mathbb{R}$ is a positive convex function on $[a,b]$ with $0 \leq a < b$ and $f \in L[a,b]$, then for $\alpha > 0$ the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

On the other hand, Íşcan in (Íşcan (2015)), proved the following Fejér type inequalities for Riemann-Liouville fractional integrals: if $f: [a,b] \longrightarrow \mathbb{R}$ is a positive convex function on $[a,b]$ with $0 \leq a < b$ and $f \in L[a,b]$, then for $\alpha > 0$ the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\left[J_{a^+}^{\alpha}(f \circ g)(b) + J_{b^-}^{\alpha}(f \circ g)(a)\right]}{\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]} \leq \frac{f(a)+f(b)}{2}, \tag{4}$$

where $g:[a,b] \longrightarrow \mathbb{R}$ be nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$ (i.e., $g(x) = g(a+b-x)$).

Let $f \in L[a,b]$. The $k$-Riemann-Liouville fractional integrals $I_{a^+}^{\alpha,k} f$ and $I_{b^-}^{\alpha,k} f$ of order $\alpha > 0$ where $k > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a$$

and

$$I_{b^-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad x < b,$$

respectively.

Let $f \in L[a,b]$. The Hadamard fractional integrals $H_{a^+}^{\alpha} f$ and $H_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$H_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} f(t) \frac{dt}{t}, \quad x > a$$

and

$$H_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} f(t) \frac{dt}{t}, \quad x < b,$$

respectively.

Let's define a function $\varphi: [0,+\infty[ \longrightarrow [0,+\infty[$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < +\infty,$$

$$\frac{1}{A_1} \leq \frac{\varphi(s)}{\varphi(r)} \leq A_1 \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,$$

$$\frac{\varphi(r)}{r^2} \leq A_2 \frac{\varphi(s)}{s^2} \quad \text{for} \quad s \leq r,$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq A_3 |r-s| \frac{\varphi(r)}{r^2} \quad \text{for} \quad \frac{1}{2} \leq \frac{s}{r} \leq 2,$$

where $A_1, A_2, A_3 > 0$ are independent of $r, s > 0$. If $\varphi(r) r^{\alpha}$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^{\beta}}$ is decreasing for some $\beta \geq 0$, then $\varphi$ satisfies the above conditions.

Sarikaya and Ertuğral in (Sarikaya Ertuğral (2019)), defined the following generalized fractional integrals:

$$I_{a^+;\varphi}f(x)=\int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt, \qquad x>a$$

and

$$I_{b^-;\varphi}f(x)=\int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt, \qquad x<b.$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integral, $k$-Riemann–Liouville fractional integral, conformable fractional integral, Hadamard fractional integrals, etc.

Motivated by the above literatures, the paper is organized as follows: In Section 3, we will establish Hermite-Hadamard-Fejér type integral inequalities for convex function involving generalized fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Also, we will derive an identity in order to develop some trapezoid type inequalities for these kind of generalized fractional integrals. Various special cases will be given and some known results will be recaptured. In Section 4, a briefly conclusion is provided as well.

## 3. Main Results

In this section, we obtain some Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals.

**Theorem 3.1.**

Let $g:[a,b]\longrightarrow\mathbb{R}$ be nonnegative and integrable function. If $f:[a,b]\longrightarrow\mathbb{R}$ is a convex function on $[a,b]$ with $a<b$, then the following Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals hold:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\frac{\left[I_{a^+;\varphi}(g\circ F)(b)+I_{b^-;\varphi}(g\circ F)(a)\right]}{\left[I_{a^+;\varphi}g(b)+I_{b^-;\varphi}g(a)\right]}\leq\frac{f(a)+f(b)}{2}, \tag{5}$$

where $F(x)=f(x)+\widetilde{f}(x)$ and $\widetilde{f}(x)=f(a+b-x)$ for all $x\in[a,b]$.

***Proof:***

Since f is a convex mapping on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2},$$

for all $x, y \in [a,b]$. For $t \in [0,1]$, let $x = ta + (1-t)b$ and $y = (1-t)a + tb$. Then, we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(ta+(1-t)b) + f\big((1-t)a+tb\big)}{2}. \tag{6}$$

Multiplying both sides of (6) by

$$\frac{(b-a)\varphi(b-[(1-t)a+tb])}{b-[(1-t)a+tb]} g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to $t$ over $(0,1)$, we obtain

$$(b-a)f\left(\frac{a+b}{2}\right)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])} g\big((1-t)a+tb\big)\,dt$$

$$\leq (b-a)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])} g\big((1-t)a+tb\big) f(ta+(1-t)b)\,dt$$

$$\leq (b-a)\int_0^1 \frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])} g\big((1-t)a+tb\big) f\big((1-t)a+tb\big)dt.$$

Using the change of variable $\tau = (1-t)a + tb$, we have

$$f\left(\frac{a+b}{2}\right) I_{a^+;\varphi} g(b) \leq \frac{1}{2} I_{a^+;\varphi}(g \circ F)(b). \tag{7}$$

Similarly, multiplying both sides of (6) by

$$\frac{(b-a)\varphi([(1-t)a+tb]-a)}{[(1-t)a+tb]-a} g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to $t$ over $(0,1)$, we get

$$f\left(\frac{a+b}{2}\right) I_{b^-;\varphi} g(a) \leq \frac{1}{2} I_{b^-;\varphi}(g \circ F)(a). \tag{8}$$

Summing the inequalities (7) and (8), we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\left[I_{a^+;\varphi}(g\circ F)(b) + I_{b^-;\varphi}(g\circ F)(a)\right]}{\left[I_{a^+;\varphi} g(b) + I_{b^-;\varphi} g(a)\right]}.$$

So, the left-side of (5) is proved. For the proof of the right-side inequality in (5), since $f$ is convex on $[a,b]$, we have

$$f(ta+(1-t)b)+f\big((1-t)a+tb\big)\leq f(a)+f(b). \tag{9}$$

Multiplying both sides of (9) by

$$\frac{(b-a)\varphi(b-[(1-t)a+tb])}{b-[(1-t)a+tb]}g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to $t$ over $(0,1)$, we get

$$\begin{aligned}&(b-a)\int_0^1\frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}g\big((1-t)a+tb\big)f(ta+(1-t)b)\,dt\\&+(b-a)\int_0^1\frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}g\big((1-t)a+tb\big)f\big((1-t)a+tb\big)\,dt\\&\leq(b-a)\big(f(a)+f(b)\big)\int_0^1\frac{\varphi(b-[(1-t)a+tb])}{(b-[(1-t)a+tb])}g\big((1-t)a+tb\big)\,dt.\end{aligned}$$

Then, we obtain

$$I_{a^+;\varphi}(g\circ F)(b)\leq\big(f(a)+f(b)\big)I_{a^+;\varphi}g(b). \tag{10}$$

Similarly, multiplying both sides of (9) by

$$\frac{(b-a)\varphi([(1-t)a+tb]-a)}{[(1-t)a+tb]-a}g\big((1-t)a+tb\big)$$

and integrating the resulting inequality with respect to $t$ over $(0,1)$, we have

$$I_{b^-;\varphi}(g\circ F)(a)\leq\big(f(a)+f(b)\big)I_{b^-;\varphi}g(a). \tag{11}$$

By adding the inequalities (10) and (11), we get

$$\frac{1}{2}\frac{\left[I_{a^+;\varphi}(g\circ F)(b)+I_{b^-;\varphi}(g\circ F)(a)\right]}{\left[I_{a^+;\varphi}g(b)+I_{b^-;\varphi}g(a)\right]}\leq\frac{f(a)+f(b)}{2},$$

which completes the proof of Theorem 3.1. ∎

**Corollary 3.1.**

Under the assumptions of Theorem 3.1, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{4}\frac{\left[I_{a^+;\varphi}F(b) + I_{b^-;\varphi}F(a)\right]}{\Phi(a,b)} \le \frac{f(a)+f(b)}{2},$$

where

$$\Phi(a,b) = \int_a^b \frac{\varphi(t-a)}{t-a}dt = \int_a^b \frac{\varphi(b-t)}{b-t}dt.$$

***Proof:***

If we choose $g(t) = 1$ in Theorem 3.1, we get the above Hermite-Hadamard type inequality for generalized fractional integrals.

**Remark 3.1.**

Taking $\varphi(t) = t$ in Corollary 3.1, then we obtain the classical Hermite-Hadamard type inequality.

**Corollary 3.2.**

Under the assumptions of Theorem 3.1, we have Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

***Proof:***

Taking $g(t) = 1$ and $\varphi(t) = \frac{1}{\Gamma(\alpha)}\frac{[\log x - \log(x-t)]^{\alpha-1}}{x-t}$, where $\alpha \in (0,1)$ in Theorem 3.1, then we get Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

**Corollary 3.3.**

Under the assumptions of Theorem 3.1, we obtain

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\frac{\int_a^b (g \circ F)(t)dt}{\int_a^b g(t)dt} \le \frac{f(a)+f(b)}{2}.$$

***Proof:***

Taking $\varphi(t) = t$ in Theorem 3.1, then we have the above inequality.

**Corollary 3.4.**

Under the assumptions of Theorem 3.1, we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\left[J_{a^+}^{\alpha}(g\circ F)(b) + J_{b^-}^{\alpha}(g\circ F)(a)\right]}{\left[J_{a^+}^{\alpha}g(b) + J_{b^-}^{\alpha}g(a)\right]} \leq \frac{f(a)+f(b)}{2}.$$

***Proof:***

Choosing $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 3.1, then we obtain the above inequality.

**Corollary 3. 5.**

Under the assumptions of Theorem 3.1, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\left[I_{a^+}^{\alpha,k}(g\circ F)(b) + I_{b^-}^{\alpha,k}(g\circ F)(a)\right]}{\left[I_{a^+}^{\alpha,k}g(b) + I_{b^-}^{\alpha,k}g(a)\right]} \leq \frac{f(a)+f(b)}{2}.$$

***Proof:***

Taking $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 3.1, then we get the above inequality.

**Corollary 3. 6.**

Under the assumptions of Theorem 3.1, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\left[\int_a^b[(g\circ F)(t) + (g\circ F)(a+b-t)]d_{\alpha}t\right]}{\int_a^b[g(t)+g(a+b-t)]d_{\alpha}t} \leq \frac{f(a)+f(b)}{2}.$$

***Proof:***

Choosing $\varphi(t) = t(b-t)^{\alpha-1}$, where $\alpha \in (0,1)$ in Theorem 3.1, then we have the above inequality.

**Corollary 3. 7.**

Under the assumptions of Theorem 3.1, we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\frac{\left[M_{g\circ F}(a,b;\alpha) + N_{g\circ F}(a,b;\alpha)\right]}{\left[M_g(a,b;\alpha) + N_g(a,b;\alpha)\right]} \leq \frac{f(a)+f(b)}{2},$$

where

$$M_g(a,b;\alpha) = \frac{1}{\alpha}\int_a^b exp\big(-A(b-t)\big)\, g(t)dt$$

and

$$N_g(a,b;\alpha) = \frac{1}{\alpha}\int_a^b exp(-A(t-a))\, g(t)dt.$$

***Proof:***

Taking $\varphi(t) = \frac{t}{\alpha} exp(-At)$, where $A = \frac{1-\alpha}{\alpha}, \alpha \in (0,1)$ in Theorem 3.1, then we obtain the above inequality.

**Lemma 3. 1.**

Let $g:[a,b] \longrightarrow \mathbb{R}$ be nonnegative and integrable function. If $f:[a,b] \longrightarrow \mathbb{R}$ is a differentiable mapping on $(a,b)$ with $a < b$, then the following identity for generalized fractional integrals holds:

$$\left[\frac{f(a)+f(b)}{2}\right]\left[I_{a^+;\varphi}g(b) + I_{b^-;\varphi}g(a)\right] - \frac{1}{2}\left[I_{a^+;\varphi}(g\circ F)(b) + I_{b^-;\varphi}(g\circ F)(a)\right]$$

$$= \frac{1}{2}\int_a^b P_{g;\varphi}(t)f'(t)dt, \tag{12}$$

where the mapping $P_{g;\varphi}: [a,b] \longrightarrow \mathbb{R}$ is defined by

$$P_{g;\varphi}(t) = \int_{a+b-t}^{t} \frac{\varphi(s-a)}{s-a} g(s)ds + \int_{a+b-t}^{t} \frac{\varphi(b-s)}{b-s} g(s)ds$$

and $F(x) = f(x) + \widetilde{f}(x)$, where $\widetilde{f}(x) = f(a+b-x)$.

***Proof:***

We denote

$$T_{f,g;\varphi}(a,b) = \int_a^b P_{g;\varphi}(t)f'(t)dt. \tag{13}$$

Integrating by parts $(13)$, we have

$$T_{f,g;\varphi}(a,b) = P_{g;\varphi}(t)f(t)\Big|_a^b - \int_a^b P'_{g;\varphi}(t)f(t)dt$$

$$= P_{g;\varphi}(b)f(b) - P_{g;\varphi}(a)f(a) - \int_a^b P'_{g;\varphi}(t)f(t)dt. \tag{14}$$

$$P_{g;\varphi}(b) = \int_a^b \frac{\varphi(s-a)}{s-a} g(s)ds + \int_a^b \frac{\varphi(b-s)}{b-s} g(s)ds$$

$$= I_{b^-;\varphi}g(a) + I_{a^+;\varphi}g(b) = -P_{g;\varphi}(a). \tag{15}$$

$$P'_{g;\varphi}(t) = \frac{\varphi(t-a)}{t-a} g(t) + \frac{\varphi(b-t)}{b-t} g(a+b-t)$$
$$+ \frac{\varphi(b-t)}{b-t} g(t) + \frac{\varphi(t-a)}{t-a} g(a+b-t).$$

$$\int_a^b P'_{g;\varphi}(t) f(t) dt = I_{a^+;\varphi}(g \circ F)(b) + I_{b^-;\varphi}(g \circ F)(a). \tag{16}$$

Then, substituting equalities (15) and (16) in (14) we get the desired equality (12). This completes the proof of Lemma 3.1. ■

Using Lemma 3.1 we can derive the following result for function, whose the first derivative in absolute value are convex using generalized fractional integrals.

**Theorem 3.2.**

Let $g: [a, b] \longrightarrow \mathbb{R}$ be nonnegative and integrable function. If $f: [a, b] \longrightarrow \mathbb{R}$ is a differentiable mapping on $(a, b)$ with $a < b$ and $|f'|$ is a convex mapping on $[a, b]$, then we have the following trapezoid type inequality for generalized fractional integrals:

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{\left[H_{g;\varphi}(a,a) + H_{g;\varphi}(b,b)\right]}{2(b-a)} [f'(a) + f'(b)], \tag{17}$$

where

$$H_{g;\varphi}(x,y) := \int_a^{\frac{a+b}{2}} \left( \int_t^{a+b-t} \frac{\varphi(y-s)}{y-s} g(s) ds \right) |x-t| dt$$
$$+ \int_{\frac{a+b}{2}}^b \left( \int_{a+b-t}^t \frac{\varphi(y-s)}{y-s} g(s) ds \right) |x-t| dt.$$

***Proof:***

Taking the modulus and using their properties in Lemma 3.1, we have

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{1}{2} \int_a^b \left|P_{g;\varphi}(t)\right| |f'(t)| dt.$$

Since $f$ is a convex mapping on $[a, b]$, we get

$$|f'(t)| = \left| f'\left( \frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| \leq \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)|.$$

Hence,

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{1}{2} \int_a^b \left|P_{g;\varphi}(t)\right| \left[ \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right] dt$$

$$= \frac{|f'(a)|}{2(b-a)} \int_a^b |P_{g;\varphi}(t)|(b-t)dt + \frac{|f'(b)|}{2(b-a)} \int_a^b |P_{g;\varphi}(t)|(t-a)dt. \tag{18}$$

Since $g$ is nonnegative function on $[a,b]$, then $P_{g;\varphi}(t)$ is nondecreasing function on $[a,b]$. As a result, we obtain

$$\begin{cases} P_{g;\varphi}(t) \le 0, & t \in \left[a, \frac{a+b}{2}\right], \\ P_{g;\varphi}(t) > 0, & t \in \left]\frac{a+b}{2}, b\right]. \end{cases}$$

Thus, it follows that

$$\begin{aligned} \int_a^b |P_{g;\varphi}(t)|(b-t)dt = & \int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \frac{\varphi(s-a)}{s-a} g(s)ds\right)(b-t)dt \\ & + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \frac{\varphi(s-a)}{s-a} g(s)ds\right)(b-t)dt \\ & + \int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} \frac{\varphi(b-s)}{b-s} g(s)ds\right)(b-t)dt \\ & + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t \frac{\varphi(b-s)}{b-s} g(s)ds\right)(b-t)dt \\ = & H_{g;\varphi}(b,a) + H_{g;\varphi}(b,b). \end{aligned}$$

Using the fact that $H_{g;\varphi}(b,a) = H_{g;\varphi}(a,a)$, we have

$$\int_a^b |P_{g;\varphi}(t)|(b-t)dt = H_{g;\varphi}(a,a) + H_{g;\varphi}(b,b). \tag{19}$$

Similarly,

$$\int_a^b |P_{g;\varphi}(t)|(t-a)dt = H_{g;\varphi}(a,a) + H_{g;\varphi}(b,b). \tag{20}$$

If we put equalities (19) and (20) in inequality (18), we obtain the desired inequality (17). The proof of Theorem 3.2 is completed. ■

**Corollary 3.8.**

Under the assumptions of Theorem 3.2, we have Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

***Proof:***

Taking $g(t) = 1$ and $\varphi(t) = \frac{1}{\Gamma(\alpha)} \frac{[\log x - \log(x-t)]^{\alpha-1}}{x-t}$, where $\alpha \in (0,1)$ in Theorem 3.2, then we get Hadamard fractional integral inequalities proved by Jleli and Samet in (Jleli and Samet (2016)).

**Corollary 3. 9.**

Under the assumptions of Theorem 3.2, we obtain

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{3K}{16}(b-a)^2[f'(a)+f'(b)].$$

***Proof:***

Let $\varphi(t) = t$ and $g(s) \leq K,\ \forall s \in [a,b]$, where $K$ is a constant in Theorem 3.2, then we have the above inequality.

**Corollary 3. 10.**

Under the assumptions of Theorem 3.2, we get

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{K[(\alpha+2)(2^{\alpha+2}-5)]}{\Gamma(\alpha+3)}\left(\frac{b-a}{2}\right)^{\alpha+2}\frac{[f'(a)+f'(b)]}{(b-a)}.$$

***Proof:***

If $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $g(s) \leq K,\ \forall s \in [a,b]$, where $K$ is a constant in Theorem 3.2, then we obtain the above inequality.

**Corollary 3. 11.**

Under the assumptions of Theorem 3.2, we have

$$\left|T_{f,g;\varphi}(a,b)\right| \leq \frac{K\left[(\alpha+2k)\left(2^{\frac{\alpha}{k}+2}-5\right)\right]}{k\Gamma_k(\alpha+3k)}\left(\frac{b-a}{2}\right)^{\frac{\alpha}{k}+2}\frac{[f'(a)+f'(b)]}{(b-a)}.$$

***Proof:***

For $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $g(s) \leq K,\ \forall s \in [a,b]$, where $K$ is a constant in Theorem 3.2, then we get the above inequality.

**Remark 3. 2.**

Taking $\varphi(t) = t(b-t)^{\alpha-1}$ or $\varphi(t) = \frac{t}{\alpha}exp(-At)$, where $A = \frac{1-\alpha}{\alpha}$ and $\alpha \in (0,1)$ in Theorem 3.2, then we get some new Hermite-Hadamard-Fejér type inequalities. In the special case where

$g(t) = 1$ in Theorem 3.2, then we can establish some new Hermite-Hadamard type inequalities. Applying our Theorems 3.1 and 3.2, for suitable options of convex function $f(x)$, for example $f(x) = x^r$, where $r > 1$ and $x > 0$; $f(x) = \frac{1}{x}$, $x > 0$, $f(x) = e^x$, $x \in \mathbb{R}$, etc., we can construct some new generalized conformable fractional integral inequalities. Also, we can obtain several new general fractional integral inequalities using special means (arithmetic, geometric, logarithmic, etc.). Some new bounds for the midpoint and trapezium quadrature formula using our results can be provided as well. We omit their proofs and the details are left to the interested reader.

## 4. Conclusion

In this work, authors established Hermite-Hadamard-Fejér type inequalities for convex function involving generalized fractional integrals with respect to another function which are generalization of some important fractional integrals such as the Riemann-Liouville fractional integrals and the Hadamard fractional integrals. Also, we obtained some trapezoid type inequalities for these kind of generalized fractional integrals. This class of convex functions, can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

## *Acknowledgments*

*The authors would like to express their heartfelt thanks to the editors and anonymous referees for their most valuable comments and constructive suggestions which leads to the significant improvement of the earlier version of the manuscript.*

## REFERENCES

Ali, A., Gulshan, G., Hussain, R., Latif, A. and Muddassar, M. (2017). Generalized inequalities of the type of Hermite-Hadamard-Fejér with quasi-convex functions by way of $k$-fractional derivatives, J. Comput. Anal. Appl., Vol. 22, No. 7, pp. 1208-1219.

Azpeitia, A.G. (1994). Convex functions and the Hadamard inequality, Rev. Colombiana Math., Vol. 28, pp. 7-12.

Bakula, M.K. and Pečarič, J. (2004). Note on some Hadamard-type inequalities, J. Inequal. Pure Appl. Math., Vol. 5, No. 3.

Bombardelli, M. and Varosaneč, S. (2009). Properties of $h$-convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl., Vol. 58, pp. 1869-1877.

Budak, H. and Sarikaya, M.Z. (2016). Hermite-Hadamard type inequalities for $s$-convex mappings via fractional integrals of a function with respect to another function, Fasc. Math., Vol. 27, pp. 25-36.

Chen, H. and Katugampola, U.N. (2017). Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals, J. Math. Anal. Appl., Vol. 446, pp. 1274-1291.

Chen, F. and Wu, S. (2014). Fejér and Hermite-Hadamard type inequalities for harmonically convex functions, J. Appl. Math., Art. 386806,

Chu, Y.-M., Khan, M.A., Khan, T.U. and Ali, T. (2016). Generalizations of Hermite-Hadamard type inequalities for *MT*-convex functions, J. Nonlinear Sci. Appl., Vol. 9, pp. 4305-4316.

Chu, Y.-M., Khan, M.A., Khan, T.U. and Khan, J. (2017). Some new inequalities of Hermite-Hadamard type for $s$-convex functions with applications, Open Math., Vol. 15, pp. 1414-1430.

Chu, Y.-M., Khan, M.A., Ali, T. and Dragomir, S.S. (2017). Inequalities for $\alpha$-fractional differentiable functions, J. Inequal. Appl., Vol. 2017, Art. ID 93, pp. 12.

Cortez, M.J.-V., Kashuri, A., Liko, R. and Hernández, J.E.-H. (2019). Some inequalities using generalized convex functions in quantum analysis, Symmetry (MDPI), Vol. 11, No. 11, pp. 14.

Dahmani, Z. (2010). On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, Ann. Funct. Anal., Vol. 1, No. 1, pp. 51-58.

Deng J. and Wang J. (2013). Fractional Hermite-Hadamard inequalities for $(\alpha, m)$-logarithmically convex functions, J. Inequal. Appl., Vol. 2013, Art. ID 364.

Dragomir, S.S. and Agarwal, R.P. (1998). Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., Vol. 11, No. 5, pp. 91-95.

Fejér, L. (1906). Über die Fourierreihen, II. Math. Naturwiss. Anz Ungar. Akad. Wiss., Vol. 24, pp. 369-390.

Iqbal, A., Khan, M.A., Ullah, S., Kashuri, A. and Chu, Y.-M. (2018). Hermite-Hadamard type inequalities pertaining conformable fractional integrals and their applications, AIP Advances, Vol. 8, pp. 1-18.

Işcan, Í. (2015). Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals, Stud. Univ. Babeş-Bolyai, Math., Vol. 60, No. 3, pp. 355-366.

Jleli, M. and Samet, B. (2016). On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function, J. Nonlinear Sci. Appl., Vol. 9, pp. 1252-1260.

Kashuri, A. and Liko, R. , (2019). Some new Hermite-Hadamard type inequalities and their applications, Stud. Sci. Math. Hung., Vol. 56, No. 1, pp. 103-142.

Kashuri A., Ramosaçaj M. and Liko R. (2020). Some new bounds of Gauss-Jacobi and Hermite-Hadamard type integral inequalities, Ukr. Math. J., in press.

Kashuri, A. and Sarikaya, M.Z. (2020). Different type parameterized inequalities for preinvex functions with respect to another function via generalized fractional integral operators and their applications, Ukr. Math. J., in press.

Khan, M.A., Ali, T., Dragomir, S.S. (2018). Hermite-Hadamard type inequalities for conformable fractional integrals, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., Vol. 112, No. 4, pp. 1033-1048.

Khan, M.A., Ali, T. and Khan, T.U. (2017). Hermite-Hadamard type inequalities with applications, Fasc. Math., Vol. 59, pp. 57-74.

Khan, M.A., Chu, Y.-M., Kashuri, A., Liko, R. and Ali, G. (2018). New Hermite-Hadamard inequalities for conformable fractional integrals, J. Funct. Spaces, Art. ID 6928130, pp. 9.

Khan, M.A., Khurshid, Y. and Ali, T. (2017). Hermite-Hadamard inequality for fractional integrals via $\eta$-convex functions, Acta Math. Univ. Comenian., Vol. 86, No. 1, pp. 153-164.

Khan, M.A., Khurshid, Y., Ali, T. and Rehman, N. (2016). Inequalities for three times differentiable functions, Punjab Univ. J. Math., Vol. 48, No. 2, pp. 35-48.

Khan, M.A., Iqbal, A., Suleman, M. and Chu, Y.-M. (2018). Hermite-Hadamard type inequalities for fractional integrals via Green function, J. Inequal. Appl., Vol. 161, pp. 1-15.

Khan, M.A., Khurshid, Y., Du, T.-S. and Chu, Y.-M. (2018). Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals, J. Funct. Spaces, Vol. 2018, Art. ID 5357463, pp. 12.

Khan, M.A., Khurshid, Y., Dragomir, S.S. and Ullah, R. (2018). Inequalities of the Hermite-Hadamard type with applications, Punjab Univ. J. Math., Vol. 50, No. 3, pp. 1-12.

Khan, M.A., Khurshid, Y. and Chu, Y.-M. (2019). Hermite-Hadamard type inequalities via the Montgomery identity, Commun. in Math. Appl., Vol. 10, No. 1, pp. 85-97.

Khurshid, Y., Khan, M.A. and Chu, Y.-M. (2019). Hermite-Hadamard-Fejér inequalities for conformable fractional integrals via preinvex functions, J. Funct. Spaces, Vol. 2019, Art. ID 3146210, pp. 9.

Khurshid, Y., Khan, M.A. and Chu, Y.-M. (2018). Generalized inequalities via *GG*-convexity and *GA*-convexity, J. Funct. Spaces, Vol. 2019, Art. ID 6926107, pp. 8.

Mubeen, S., Iqbal, S. and Tomar, M. (2016). On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function and $k$-parameter, J. Inequal. Math. Appl., Vol. 1, pp. 1-9.

Peng, S., Wei, W. and Wang, J-R. (2014). On the Hermite-Hadamard inequalities for convex functions via Hadamard fractional integrals, Facta Univ., Ser. Math. Inf., Vol. 29, No. 1, pp. 55-75.

Rassias, T.M. and Kashuri, A. (2019). Some new bounds of Gauss-Jacobi and Hermite-Hadamard type integral inequalities, Appl. Anal. Discrete Math., in press.

Sarikaya, M.Z. (2012). On new Hermite-Hadamard-Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai, Math., Vol. 57, No. 3, pp. 377-386.

Sarikaya, M.Z. and Budak, H. (2017). On Fejér type inequalities via local fractional integrals, J. Fract. Calc. Appl., Vol. 8, No. 1, pp. 59-77.

Sarikaya, M.Z. and Budak, H. (2016). Generalized Hermite-Hadamard type integral inequalities for fractional integrals, Filomat, Vol. 30, No. 5, pp. 1315-1326.

Sarikaya, M.Z. and Budak, H. (2014). Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals, Facta Univ., Ser. Math. Inform., Vol. 29, No. 4, pp. 371-384.

Sarikaya, M.Z. and Ertuğral, F. (2019). On the generalized Hermite-Hadamard inequalities, Annals of the University of Craiova–Mathematics and Computer Science Series, accepted.

Sarikaya, M.Z., Yaldiz, H. and Erden, S. (2014). Some inequalities associated with the Hermite-Hadamard-Fejér type for convex function, Mathematical Sciences, Vol. 8, pp. 117-124.

Sarikaya, M.Z., Set, E., Yaldiz, H. and Basak, N. (2013). Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, Math. Comput. Modelling, Vol. 57, pp. 2403-2407.

Set, E., Işcan, Í., Sarikaya, M.Z. and Özdemir, M.E. (2015). On new inequalities of Hermite-Hadmard-Fejér type for convex functions via fractional integrals, Appl. Math. Comput., Vol. 259, No. 1, pp. 875-881.

Set, E., Sarikaya, M.Z., Özdemir, M.E. and Yildirim, H. (2014). The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results, J. Appl. Math. Stat. Inform., Vol. 10, No. 2, pp. 69-83.

Set, E., Karatas, S.S. and Khan, M.A. (2016). Hermite-Hadamard type inequalities obtained via fractional integral for differentiable $m$-convex and $(\alpha, m)$-convex function, Int. J. Anal., Art. ID 4765691, pp. 8.

Tseng, K.-L., Yang, G.-S. and Hsu, K.-C. (2011). Some inequalities for differentiable mappings and applications to Fejér inequality and weighted trapezoidal formula, Taiwanese J. Math., Vol. 15, No. 4, pp. 1737-1747.

Wang, J.R., Li, X. and Zhu, C. (2013). Refinements of Hermite-Hadamard type inequalities involving fractional integrals, Bull. Belg. Math. Soc. Simon Stevin, Vol. 20, pp. 655-666.

Wang, J.R., Zhu, C. and Zhou, Y. (2013). New generalized Hermite-Hadamard type inequalities and applications to special means, J. Inequal. Appl., Vol. 2013, No. 325.

Zhang, Y. and Wang, J. (2013). On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, J. Inequal. Appl., Vol. 2013, Art. ID 220.

Zhang, Z., Wei, W. and Wang, J. (2015). Generalization of Hermite-Hadamard inequalities involving Hadamard fractional integrals, Filomat, Vol. 29, No. 7, pp. 1515-1524.