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## On Generalized Hurwitz-Lerch Zeta Distributions

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### Abstract

In this paper, we introduce a function  $\Phi_{\alpha,\beta;\gamma}(z,s,a)$ , which is an extension to the general Hurwitz-Lerch Zeta function. Having defined the incomplete generalized beta type-2 and incomplete generalized gamma functions, some differentiation formulae are established for these incomplete functions. We have introduced two new statistical distributions, termed as generalized Hurwitz-Lerch Zeta beta type-2 distribution and generalized Hurwitz-Lerch Zeta gamma distribution and then derived the expressions for the moments, distribution function, the survivor function, the hazard rate function and the mean residue life function for these distributions. Graphs for both these distributions are given, which reflect the role of shape and scale parameters.

**Keywords:** Riemann Zeta Function; Lerch Zeta Function; General Hurwitz-Lerch Zeta Function; Gauss Hypergeometric Function; Beta Type-2 Distribution; Gamma Distribution; Plank Distribution

**MSC (2000) No.:** 11M35, 60E05

## 1. Introduction

A general Hurwitz-Lerch Zeta function  $\Phi(z,s,a)$  is defined in the following manner in the book by Erdelyi et al. (1953)

$$\Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad (1.1)$$

( $a \neq 0, -1, -2, \dots, s \in C$ , when  $|z| < 1$  and  $\text{Re}(s) > 1$ , when  $|z| = 1$ ). It contains, as its special cases, the Riemann Zeta function and Hurwitz Zeta function, defined as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1), \quad \text{Re}(s) > 1 \quad (1.2)$$

and

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a), \quad \text{Re}(s) > 1, \quad a \neq 0, -1, -2, \dots \quad (1.3)$$

An extension to the general Hurwitz-Lerch Zeta function (1.1) is defined in the series form, as

$$\Phi_{\alpha,\beta;\gamma}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \frac{z^n}{(n+a)^s}, \quad (1.4)$$

( $\gamma, a \neq 0, -1, -2, \dots, s \in C$ , when  $|z| < 1$  and  $\text{Re}(\gamma + s - \alpha - \beta) > 0$ , when  $|z| = 1$ ), and equivalently in the integral form, as

$$\Phi_{\alpha,\beta;\gamma}(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-at} {}_2F_1(\alpha, \beta; \gamma; ze^{-t}) dt, \quad (1.5)$$

( $\text{Re}(a) > 0; \gamma \neq 0, -1, -2, \dots, \text{Re}(s) > 0$  and  $|z| < 1$  or  $|z| = 1$  with  $\text{Re}(\gamma - \alpha - \beta) > 0$ ), where  ${}_2F_1(\alpha, \beta; \gamma; z)$  is the Gauss hypergeometric function defined in Rainville (1960). We can easily obtain another integral representation of  $\Phi_{\alpha,\beta;\gamma}(z,s,a)$ , given by

$$\Phi_{\alpha,\beta;\gamma}(z,s,a) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^{\infty} t^{\beta-1} (1+t)^{-\gamma} \Phi_{\alpha}^*\left(\frac{tz}{1+t}, s, a\right) dt, \quad (1.6)$$

( $\text{Re}(\beta) > 0, \text{Re}(\gamma - \beta) > 0, a \neq 0, -1, -2, \dots; s \in C$ , when  $|z| < 1$  and  $\text{Re}(s - \alpha) > 0$ , when  $|z| = 1$ ), where

$$\Phi_{\alpha}^{*}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \frac{z^n}{(n+a)^s} \quad (1.7)$$

( $a \neq 0, -1, -2, \dots, s \in C$ , when  $|z| < 1$  and  $\operatorname{Re}(s - \alpha) > 0$ , when  $|z| = 1$ ), is generalized Hurwitz-Lerch Zeta function, defined by Goyal and Laddha (1997).

## Special Cases

(i) If we take  $\beta = 1$  in (1.4), we arrive at a generalization of Hurwitz-Lerch Zeta function, defined by Lin and Srivastava (2004)

$$\Phi_{\alpha,1;\gamma}(z, s, a) = \Phi_{\alpha,\gamma}^{(1,1)}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{(n+a)^s}, \quad (1.8)$$

( $\gamma, a \neq 0, -1, \dots; s \in C$ , when  $|z| < 1$  and  $\operatorname{Re}(\gamma + s - \alpha) > 1$ , when  $|z| = 1$ ).

(ii) If we take  $\beta = \gamma$  in (1.4), we get (1.7).

(iii) Taking  $\beta = \gamma$  and  $\alpha = 1$  in (1.4), we get general Hurwitz-Lerch Zeta function (1.1).

## 2. The Generalized Incomplete Functions

The incomplete generalized beta type-2 function is defined by

$$B_{\alpha,\beta;\gamma}^{0,x}(z, s, a) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^x t^{\beta-1} (1+t)^{-\gamma} \Phi_{\alpha}^{*}\left(\frac{tz}{1+t}, s, a\right) dt \quad (2.1)$$

( $\operatorname{Re}(\beta) > 0, a \neq 0, -1, -2, \dots; s \in C$ , when  $|z| < 1$  and  $\operatorname{Re}(s - \alpha) > 0$ , when  $|z| = 1$ ) and the complementary incomplete generalized beta type-2 function is

$$B_{\alpha,\beta;\gamma}^{x,\infty}(z, s, a) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_x^{\infty} t^{\beta-1} (1+t)^{-\gamma} \Phi_{\alpha}^{*}\left(\frac{tz}{1+t}, s, a\right) dt, \quad (2.2)$$

( $\operatorname{Re}(\gamma - \beta) > 0, a \neq 0, -1, -2, \dots, s \in C$ , when  $|z| < 1$  and  $\operatorname{Re}(s - \alpha) > 0$ , when  $|z| = 1$ ). We, thus, have

$$\Phi_{\alpha,\beta;\gamma}(z, s, a) = B_{\alpha,\beta;\gamma}^{0,x}(z, s, a) + B_{\alpha,\beta;\gamma}^{x,\infty}(z, s, a). \quad (2.3)$$

The incomplete generalized gamma function is defined by

$$\Gamma_{\alpha,\beta;\gamma}^{0,x}(z,s,a,b) = \frac{b^s}{\Gamma(s)} \int_0^x t^{s-1} e^{-at} {}_2F_1(\alpha,\beta;\gamma;ze^{-bt}) dt, \quad (2.4)$$

( $\gamma \neq 0, -1, -2, \dots$ ;  $\text{Re}(s) > 0$ ;  $|z| < 1$  or  $|z| = 1$ , with  $\text{Re}(\gamma - \alpha - \beta) > 0$ ), whereas the complementary incomplete generalized gamma function is given by

$$\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b) = \frac{b^s}{\Gamma(s)} \int_x^\infty t^{s-1} e^{-at} {}_2F_1(\alpha,\beta;\gamma;ze^{-bt}) dt, \quad (2.5)$$

( $\text{Re}(a) > 0, \text{Re}(b); \gamma \neq 0, -1, -2, \dots$ ,  $|z| < 1$  or  $|z| = 1$ , with  $\text{Re}(\gamma - \alpha - \beta) > 0$ ). It can be easily verified that

$$\Phi_{\alpha,\beta;\gamma}(z,s,\frac{a}{b}) = \Gamma_{\alpha,\beta;\gamma}^{0,x}(z,s,a,b) + \Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b). \quad (2.6)$$

### 3. Differentiation Formulae

Performing differentiation under the sign of integration by Leibnitz rule in equations (2.1), (2.2), (2.4) and (2.5), we obtain the following differentiation formulae for the incomplete generalized beta type-2 functions and the incomplete generalized gamma functions

$$\frac{d}{dx} [x^{1-\beta} B_{\alpha,\beta;\gamma}^{0,x}(z,s,a)] = (1-\beta)x^{-\beta} B_{\alpha,\beta;\gamma}^{0,x}(z,s,a) + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} (1+x)^{-\gamma} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right), \quad (3.1)$$

$$\frac{d}{dx} [(1+x)^{\gamma} B_{\alpha,\beta;\gamma}^{0,x}(z,s,a)] = \gamma(1+x)^{\gamma-1} B_{\alpha,\beta;\gamma}^{0,x}(z,s,a) + \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} x^{\beta-1} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right), \quad (3.2)$$

$$\frac{d}{dx} [x^{1-\beta} B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a)] = (1-\beta)x^{-\beta} B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a) - \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} (1+x)^{-\gamma} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right), \quad (3.3)$$

$$\frac{d}{dx} [(1+x)^{\gamma} B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a)] = \gamma(1+x)^{\gamma-1} B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a) - \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} x^{\beta-1} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right), \quad (3.4)$$

$$\frac{d}{dx} [x^{1-s} \Gamma_{\alpha,\beta;\gamma}^{0,x}(z,s,a,b)] = (1-s)x^{-s} \Gamma_{\alpha,\beta;\gamma}^{0,x}(z,s,a,b) + \frac{b^s}{\Gamma(s)} e^{-ax} {}_2F_1(\alpha,\beta;\gamma;ze^{-bx}), \quad (3.5)$$

$$\frac{d}{dx}[e^{ax}\Gamma_{\alpha,\beta;\gamma}^{0,x}(z,s,a,b)] = ae^{ax}\Gamma_{\alpha,\beta;\gamma}^{0,x}(z,s,a,b) + \frac{b^s}{\Gamma(s)}x^{s-1}{}_2F_1(\alpha,\beta;\gamma;ze^{-bx})\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad (3.6)$$

$$\frac{d}{dx}[x^{1-s}\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)] = (1-s)x^{-s}\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b) - \frac{b^s}{\Gamma(s)}e^{-ax}{}_2F_1(\alpha,\beta;\gamma;ze^{-bx}), \quad (3.7)$$

$$\frac{d}{dx}[e^{ax}\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)] = ae^{ax}\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b) - \frac{b^s}{\Gamma(s)}x^{s-1}{}_2F_1(\alpha,\beta;\gamma;ze^{-bx}). \quad (3.8)$$

#### 4. The Generalized Hurwitz-Lerch Zeta Beta type-2 Distribution

Special functions play a significant role in the study of probability density functions (*pdf*) see, for example, Lebedev (1965), Mathai and Saxena (1973, 1978) Mathai (1993), Johnson and Kotz [1970(a), 1970(b)] etc. Some well-known beta type-2 distributions are beta type-2 noncentral, beta type-2 inverted, beta type-2 three parameter, beta type-2 associated with chi square, Pareto distribution of 2<sup>nd</sup> kind and Fisher's F distribution. A generalization of F distribution is defined and studied by Malik (1967). More distributions of beta type-2 have been defined and studied by Mathai and Saxena (1971), Garg and Gupta (1997) and Ben-Nakhi and Kalla (2002, 2003), involving certain special functions.

The generalized Hurwitz-Lerch Zeta beta type-2 *pdf* of a random variable  $x$  is defined as

$$f(x) = \begin{cases} \frac{\Gamma(\gamma) x^{\beta-1} (1+x)^{-\gamma} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right)}{\Gamma(\beta)\Gamma(\gamma-\beta) \Phi_{\alpha,\beta;\gamma}(z, s, a)}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases} \quad (4.1)$$

where  $\Phi_{\alpha}^*(z, s, a)$  and  $\Phi_{\alpha,\beta;\gamma}(z, s, a)$  are defined by (1.7) and (1.4), respectively, together with following additional conditions:

- (i)  $\gamma > \beta > 0$ ,  $a \neq 0, -1, \dots$ ;  $s \in R$ , when  $|z| < 1$  and  $s - \alpha > 0$ , when  $|z| = 1$ .
- (ii) The parameters involved in (4.1) are so restricted that  $f(x)$  remains non-negative for  $x > 0$ .

Here,  $\beta$  and  $\gamma$  are shape parameters, whereas  $z$  represents the scale parameter.

It is easy to verify that  $\int_{-\infty}^{\infty} f(x) dx = 1$ .

We observe that the behavior of  $f(x)$  at  $x = 0$  depends on  $\beta$ , i.e.,

$$f(0) = \begin{cases} 0, & \beta > 1 \\ (\gamma - 1) [\Phi_{\alpha, \gamma}^{(1,1)}(z, s, a)]^{-1}, & \beta = 1 \\ \infty, & 0 < \beta < 1 \end{cases}$$

and  $\lim_{x \rightarrow \infty} f(x) = 0$ . By logarithmic differentiation of (4.1), we get

$$f'(x) = \left[ \frac{\beta - 1}{x} - \frac{\gamma}{1+x} + \frac{\alpha z}{(1+x)^2} \frac{\Phi_{\alpha+1}^*\left(\frac{xz}{1+x}, s, a+1\right)}{\Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right)} \right] f(x). \quad (4.2)$$

### Special Cases

(i) If we substitute  $\beta = 1$  in (4.1), we get a new probability distribution

$$f(x) = \frac{(\gamma - 1) (1+x)^{-\gamma} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right)}{\Phi_{\alpha, \gamma}^{(1,1)}(z, s, a)}, \quad x > 0$$

( $\gamma - 1 > 0, a \neq 0, -1, \dots; s \in R$ , when  $|z| < 1$  and  $s - \alpha > 0$ , when  $|z| = 1$ ), where  $(z, s, a)$  is given by (1.8).

(ii) If we take  $\gamma = \alpha$  in (4.1), we get another new probability density function

$$f(x) = \frac{\Gamma(\alpha) x^{\beta-1} (1+x)^{-\alpha} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right)}{\Gamma(\beta)\Gamma(\alpha - \beta) \Phi_{\beta}^*(z, s, a)}, \quad x > 0,$$

( $\alpha > \beta > 0, a \neq 0, -1, \dots; s \in R$ , when  $|z| < 1$  and  $s - \alpha > 0$ , when  $|z| = 1$ ).

(iii) On taking  $\alpha = 0$  in (4.1), we get beta distribution of second kind.

Fisher's F distribution, which is a beta type-2 distribution with  $x$  replaced by  $mx/n$ ,  $\beta = m/2$  and  $\gamma = (m + n)/2$  where  $m, n$  are positive integer can also be obtained from (4.1).

Graphs of the probability density function  $f(x)$ , as defined in (4.1), are represented in Figures 1 and 2. We take two values of  $z$  as 0.1 and 1 and plot the graphs for different values of the shape parameters  $\beta$  and  $\gamma$  while fixing the other parameters in Figures 1 and 2, respectively.

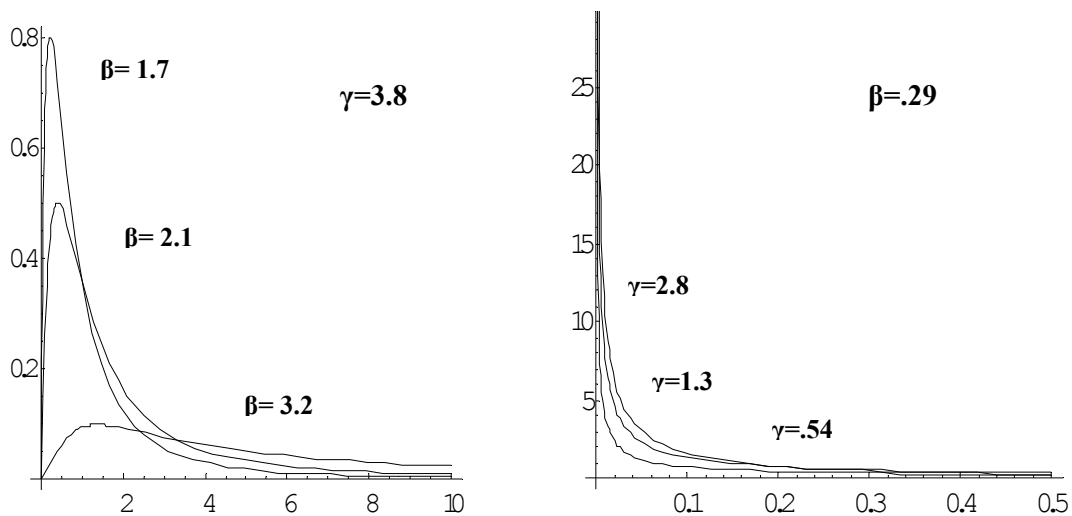


Figure 1.  $\alpha = .21, s = 2, a = .92, z = .1$

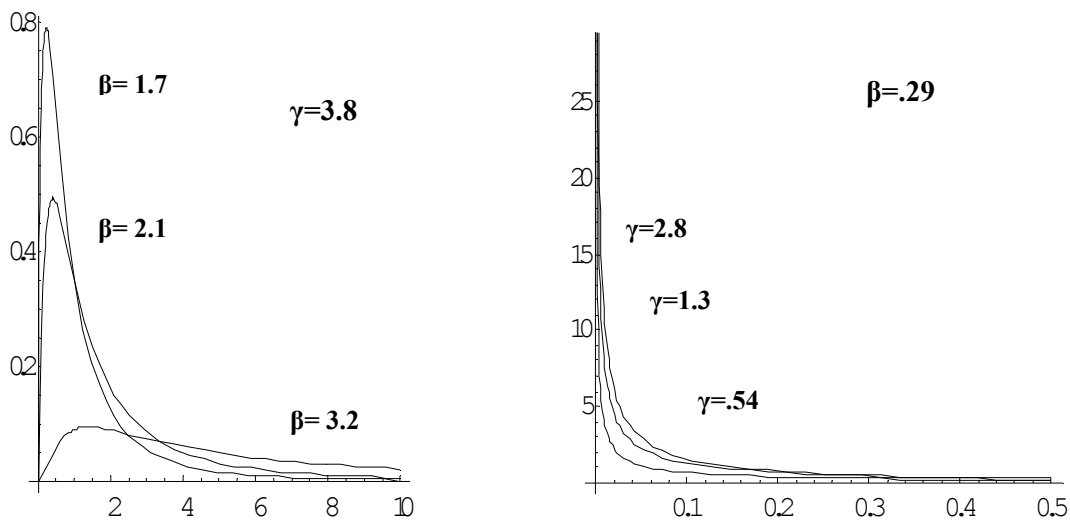


Figure 2.  $\alpha = .21, s = 2, a = .92, z = 1$

The mathematical expectation of any function  $g(x)$  with respect to pdf  $f(x)$  is given by

$$E[g(x)] = \int_{-\infty}^{\infty} f(x) g(x) dx = \int_0^{\infty} f(x) g(x) dx, \tag{4.3}$$

where  $f(x)$  is defined by (4.1).



## Moments

If we take  $g(x) = x^k$  in (4.3), we get the  $k^{th}$  moment about the origin as follows

$$E(x^k) = \int_0^{\infty} x^k f(x) dx = \frac{(-1)^k (\beta)_k}{(1-\gamma+\beta)_k} \frac{\Phi_{\alpha,\beta+k;\gamma}(z,s,a)}{\Phi_{\alpha,\beta;\gamma}(z,s,a)}. \quad (4.4)$$

Further, if we take  $g(x) = x^{t-1}$ ,  $e^{-tx}$  and  $e^{\omega tx}$  in (4.3) successively, we obtain the Mellin Transform, Laplace Transform and Fourier Transform (Characteristic function) of the *pdf*  $f(x)$ , respectively, as follows

$$E(x^{t-1}) = M[f(x);t] = \int_0^{\infty} x^{t-1} f(x) dx = \frac{(-1)^{t-1} (\beta)_{t-1}}{(1-\gamma+\beta)_{t-1}} \frac{\Phi_{\alpha,\beta+t-1;\gamma}(z,s,a)}{\Phi_{\alpha,\beta;\gamma}(z,s,a)}, \quad (4.5)$$

$$E(e^{-tx}) = L[f(x);t] = \int_0^{\infty} e^{-tx} f(x) dx = \frac{1}{\Phi_{\alpha,\beta;\gamma}(z,s,a)} \sum_{k=0}^{\infty} \frac{(\beta)_k t^k}{(1-\gamma+\beta)_k k!} \Phi_{\alpha,\beta+k;\gamma}(z,s,a), \quad (4.6)$$

$$E(e^{\omega tx}) = F[f(x);t] = \int_0^{\infty} e^{\omega tx} f(x) dx = \frac{1}{\Phi_{\alpha,\beta;\gamma}(z,s,a)} \sum_{k=0}^{\infty} \frac{(\beta)_k (-\omega t)^k}{(1-\gamma+\beta)_k k!} \Phi_{\alpha,\beta+k;\gamma}(z,s,a), \quad (4.7)$$

where

$$\omega = \sqrt{-1}.$$

## The Distribution Function

The distribution function (or cumulative distribution function) for the *pdf*  $f(x)$  is given by

$$F(x) = \int_{-\infty}^x f(t) dt = \int_0^x f(t) dt = \frac{B_{\alpha,\beta;\gamma}^{0,x}(z,s,a)}{\Phi_{\alpha,\beta;\gamma}(z,s,a)}, \quad (4.8)$$

where  $B_{\alpha,\beta;\gamma}^{0,x}(z,s,a)$  is given by (2.1).

## The Survivor Function

The Survivor function is expressed as

$$S(x) = 1-F(x) = \int_x^{\infty} f(t) dt = \frac{B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a)}{\Phi_{\alpha,\beta;\gamma}(z,s,a)}, \quad (4.9)$$

where  $B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a)$  is given by (2.2).

### The Hazard Rate Function

The Hazard rate function (or failure rate) is defined as  $h(x) = \frac{f(x)}{S(x)}$  and it can be expressed using equations (4.1) and (4.9), as

$$h(x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \frac{x^{\beta-1}(1+x)^{-\gamma} \Phi_{\alpha}^*\left(\frac{xz}{1+x}, s, a\right)}{B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a)}. \quad (4.10)$$

### The Mean Residue Life Function

For a random variable  $x$ , the mean residue life function is defined by

$$K(x) = E[X - x | X \geq x] = \frac{1}{S(x)} \int_x^{\infty} (t-x)f(t) dt,$$

which can be written in the following form with the help of equations (2.2) and (4.9)

$$K(x) = \frac{\beta B_{\alpha,\beta+1;\gamma}^{x,\infty}(z,s,a)}{(\gamma-\beta-1) B_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a)} - x. \quad (4.11)$$

## 5. The Generalized Hurwitz-Lerch Zeta Gamma Distribution

A lot of work is done by various research workers in the study of gamma type distribution involving certain special functions notably Stacy (1962), Saxena and Dash (1979), Kalla et al. (2001), Ali et al. (2001).

In the present paper we define and study a new probability density function which generalizes both the well known gamma distribution and Plank distribution given in Johnson and Kotz (1970(b)). We consider the following definition of the generalized Hurwitz-Lerch Zeta gamma distribution.

$$f(x) = \begin{cases} \frac{b^s}{\Gamma(s)\Phi_{\alpha,\beta;\gamma}\left(z, s, \frac{a}{b}\right)} x^{s-1} e^{-ax} {}_2F_1(\alpha, \beta; \gamma; z e^{-bx}), & x > 0 \\ 0, & \text{elsewhere.} \end{cases} \quad (5.1)$$

where  $\Phi_{\alpha,\beta;\gamma}(z, s, a)$  is defined by (1.4) and the following additional conditions are satisfied

- (i)  $a > 0, b > 0; \gamma \neq 0, -1, -2, \dots, s > 0$  and  $|z| < 1$  or  $|z|=1$  with  $\gamma - \alpha - \beta > 0$ .
- (ii) The parameters involved in (5.1) are so restricted that  $f(x)$  remains non-negative for  $x > 0$ .

Here,  $a$  and  $b$  represent scale parameters while  $s$  is shape parameter.

It is easy to verify that  $\int_{-\infty}^{\infty} f(x) dx = 1$

We observe that behavior of  $f(x)$  at  $x=0$  depends on  $s$ , i.e. ,

$$f(0) = \begin{cases} 0, & s > 1 \\ \frac{b {}_2F_1(\alpha, \beta; \gamma; z)}{\Phi_{\alpha,\beta;\gamma}\left(z, 1, \frac{a}{b}\right)}, & s = 1 \\ \infty, & 0 < s < 1 \end{cases}$$

and  $\lim_{x \rightarrow \infty} f(x) = 0$ . By Logarithmic differentiation of (5.1), we get

$$f'(x) = \left[ \frac{s-1}{x} - a - \frac{b\alpha\beta z e^{-bx}}{\gamma} \frac{{}_2F_1(\alpha+1, \beta+1; \gamma+1; z e^{-bx})}{{}_2F_1(\alpha, \beta; \gamma; z e^{-bx})} \right] f(x). \quad (5.2)$$

- (i) If we take  $\beta = 1$  in (5.1), then we get a new probability distribution as follows

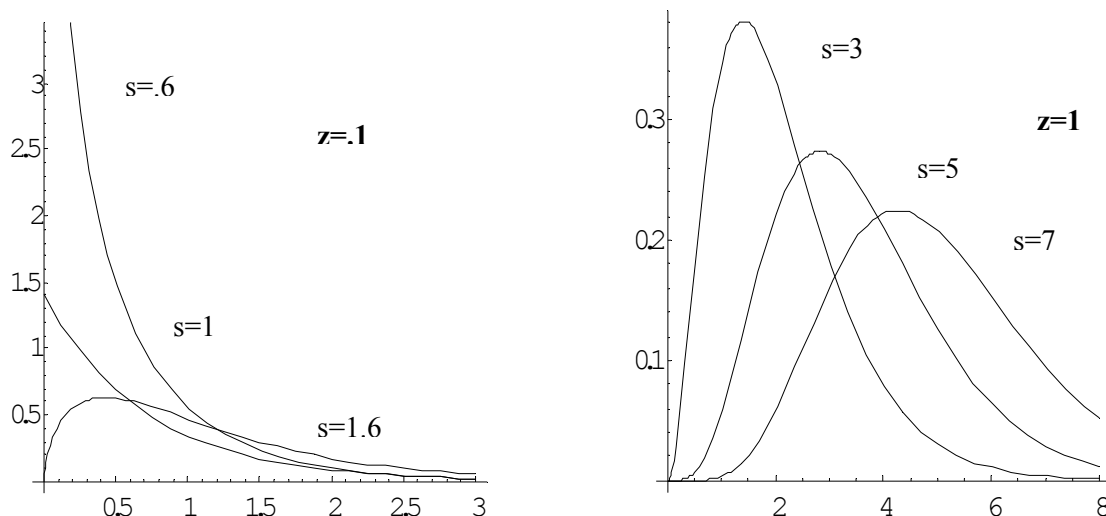
$$f(x) = \frac{b^s}{\Gamma(s)\Phi_{\alpha,\gamma}^{(1,1)}\left(z, s, \frac{a}{b}\right)} x^{s-1} e^{-ax} {}_2F_1(\alpha, 1; \gamma; z e^{-bx}), \quad x > 0$$

( $a > 0, b > 0, s > 0$  and  $|z| < 1$  or  $|z|=1$  with  $\gamma - \alpha > 1$ ), where  $\Phi_{\alpha,\gamma}^{(1,1)}(z, s, a)$  is given by (1.8).

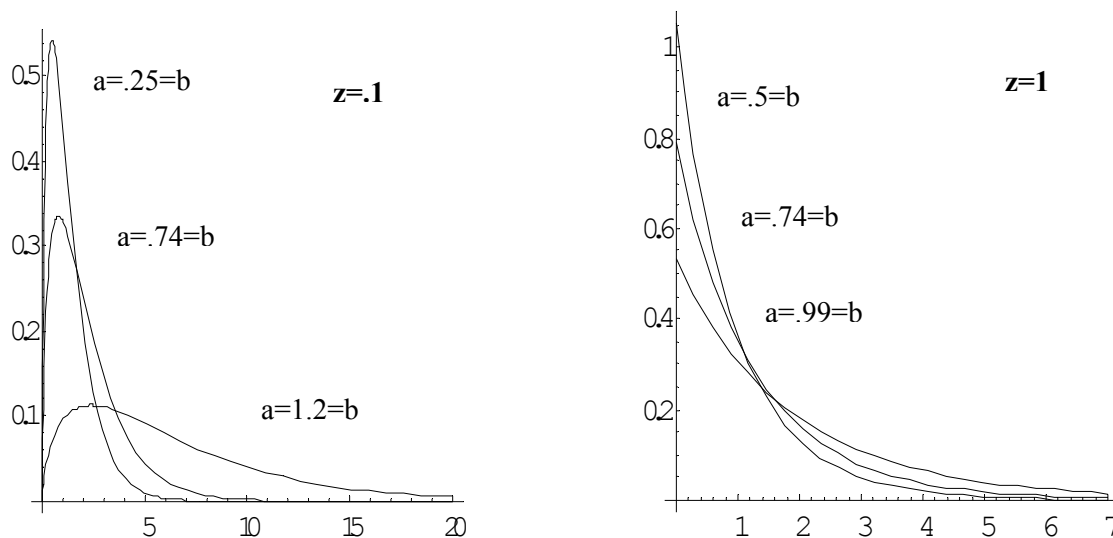
- (ii) If we set  $b = a$  and  $\alpha = 0$  in (5.1), then it reduces into well known gamma distribution.

- (iii) If we take  $\beta = \gamma$  in (5.1), then we get the unified Plank distribution, defined by Goyal and Prajapat.
- (iv) On further setting  $\alpha = 1$ , we get the generalized Plank distribution, defined by Nadarajah and Kotz (2006), which is a generalization of the well known defined in the book by Johnson and Kotz (1970(b), p.273)

The probability density function  $f(x)$  is represented in Figure 3 and 4. The effect of the shape parameter 's' for  $z=1$  and  $z=1$  is shown in Figure 3. The case, where scale parameters  $a$  and  $b$  are equal for two different values of  $z$  while fixing other parameters, is shown in Figure 4.



**Figure 3.**  $\alpha = .75, \beta = .33, \gamma = 3.2, a = 1.4, b = .5$



**Figure 4.**  $\alpha = .75, \beta = .33, \gamma = 3.2, s = 1.6$

Now, we will obtain moments, the distribution function, the survivor function  $S(x)$ , the hazard rate function  $h(x)$  and the mean residue life function  $K(x)$  for the *pdf* defined by (5.1) on the lines similar to Section 4.

### Moments

$$E(x^k) = \int_0^{\infty} x^k f(x) dx = \frac{(s)_k}{b^k} \frac{\Phi_{\alpha,\beta;\gamma}(z, s+k, \frac{a}{b})}{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a}{b})}, \quad (5.3)$$

Further, we obtain the Mellin Transform, Laplace Transform and Fourier Transform (Characteristic function) of the *pdf*  $f(x)$  as follows

$$E(x^{t-1}) = M[f(x); t] = \int_0^{\infty} x^{t-1} f(x) dx = \frac{(s)_{t-1}}{b^{t-1}} \frac{\Phi_{\alpha,\beta;\gamma}(z, s+t-1, \frac{a}{b})}{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a}{b})}, \quad (5.4)$$

$$E(e^{-tx}) = L[f(x); t] = \int_0^{\infty} e^{-tx} f(x) dx = \frac{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a+t}{b})}{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a}{b})}, \quad (5.5)$$

$$E(e^{i\omega x}) = F[f(x); t] = \int_0^{\infty} e^{i\omega x} f(x) dx = \frac{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a-i\omega t}{b})}{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a}{b})}, \quad \omega = \sqrt{-1}. \quad (5.6)$$

### The Distribution Function

$$F(x) = \frac{\Gamma_{\alpha,\beta;\gamma}^{0,x}(z, s, a, b)}{\Phi_{\alpha,\beta;\gamma}(z, s, \frac{a}{b})}, \quad (5.7)$$

where  $\Gamma_{\alpha,\beta;\gamma}^{0,x}(z, s, a, b)$  is the incomplete generalized gamma function given by (2.4).

## The Survivor Function

$$S(x) = \frac{\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)}{\Phi_{\alpha,\beta;\gamma}(z,s,\frac{a}{b})}, \quad (5.8)$$

where  $\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)$  is the complementary incomplete generalized gamma function given by (2.5).

## The Hazard Rate Function

$$h(x) = \frac{b^s x^{s-1} e^{-ax} {}_2F_1(\alpha, \beta; \gamma; ze^{-bx})}{\Gamma(s) \Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)}. \quad (5.9)$$

## The Mean Residue Life Function

$$K(x) = \frac{s}{b} \frac{\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s+1,a,b)}{\Gamma_{\alpha,\beta;\gamma}^{x,\infty}(z,s,a,b)} - x. \quad (5.10)$$

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