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Homotopy Perturbation Method and Padé Approximants for Solving Flierl-Petviashivili Equation

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Abstract

In this paper, we present a reliable combination of homotopy perturbation method and Padé approximants to investigate the Flierl-Petviashivili (FP) equation. The approach introduces a new transformation necessary for the conversion of the Flierl-Petviashivili equation to a first order initial value problem and a reliable framework designed to overcome the difficulty of the singular point at $x = 0$. The proposed homotopy perturbation method is applied to the reformulated first order initial value problem which leads the solution in terms of transformed variable. The desired series solution is obtained by making use of the inverse transformation. The suggested algorithm may be considered as an efficient and reliable scheme for solving Flierl-Petviashivili equation and other singular boundary value problems.

Keywords Flierl-Petviashivili equation, homotopy perturbation method, Padé approximants

AMS Class: 65 N 10, 34 Bxx

1. Introduction

In the last two decades with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists, physicists and engineers in the analytical techniques for nonlinear problems. It is well known, that perturbation methods provide the most versatile tools available in nonlinear analysis of engineering problems, see He ((1999), (2000), (2004), (2005), (2006), (2008)), Noor and Mohyud-Din ((2006), (2007), (2008)). The Perturbation methods, like other nonlinear analytical techniques, have their own limitations.

At first, almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to the ideal results but, an unsuitable choice may create serious problems. Furthermore, the approximate solutions solved by perturbation methods are valid, in most cases, only for the small values of the parameters.

It is obvious that all these limitations come from the small parameter assumption. These facts have motivated to suggest alternate techniques such as, the variational iteration method (see He (2006), (2007), Noor and Mohyud-Din (2006), (2007), (2008)), decomposition method (see Wazwaz (2006), exp-function method (see Noor, Mohyud-Din and Waheed (2008)) and the iterative method (see Noor and Mohyud-Din (2006), (2007)). In order to overcome these drawbacks, combining the standard homotopy method and perturbation, which is called the homotopy perturbation, modifies the homotopy method. In this paper, we implement the modified homotopy perturbation method developed for solving Flierl-Petviashivili (FP) equation.

The FP equation plays a very important role in thermodynamics, mathematical physics and astrophysics, theory of stellar structure, thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents, (see Adomian (1992), (1995), Chandrasekhar (1967), Davis (1962), Russel (1975), Shawagfeh (1993), Wazwaz (2006), Noor and Mohyud-Din (2007)). It is shown that the proposed modification plays a fundamental and pivotal role in the implementation of homotopy perturbation method. The approach introduces a new transformation necessary for the conversion of the Flierl-Petviashivili (FP) equation to a first order initial value problem and a reliable framework designed to overcome the difficulty of the singular point at $x = 0$. The proposed homotopy perturbation method is applied to the reformulated first order initial value problem which leads the solution in terms of transformed variable.

The desired series solution is obtained by making use of the inverse transformation. The suggested algorithm may be considered as an efficient and reliable scheme for solving FP equation and other singular boundary value problems. This technique is easy to implement and is more efficient than the Adomian's decomposition method. The fact that the proposed method solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the Adomian's decomposition method.

2. Homotopy perturbation method

The essence of the homotopy perturbation method is the introduction of the homotopy parameter p which takes the value from 0 to 1. When $p = 0,$ the equation or system of equations takes a simplified form whose solution can be readily obtained analytically. As p is increased and eventually takes the value 1, the system of equations evolves to the required form, and it is expected that the solution will approach the desired value. The variables of interest are expresses as a power series in p, see He (1999), (2000), (2004), (2005), (2006), Noor and Mohyud-Din (2006), (2007), (2008). Now, to explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{1}$$

where L is any integral or differential operator. We define a convex homotopy H (u, p) by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{3}$$

we have

$$H(u,0) = F(u), \qquad H(u,1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(f, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter (see He (1999), (2000), (2004), (2005), (2006), Noor and Mohyud-Din (2006), (2007), (2008)). The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{4}$$

if p $\rightarrow$ 1, then (4) corresponds to (2) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{5}$$

It is well known that series (5) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$; see He (1999), (2000), (2004), (2005), (2006). We assume that (5) has a unique solution. The comparisons of like powers of p give solutions of various orders.

3. Padé Approximants

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[L/M]$ Padé approximants to a function $y(x)$ are given by Boyd (1997), Momani (2007)

$$\left[\frac{L}{M}\right]=\frac{P_L(x)}{Q_M(x)}, \tag{6}$$

where $P_L(x)$ is polynomial of degree at most L and $Q_M(x)$ is a polynomial of degree at most M. the formal power series

$$y(x)=\sum_{i=1}^{\infty}a_i x^i, \tag{7}$$

$$y(x)-\frac{P_L(x)}{Q_M(x)}=O(x^{L+M+1}), \tag{8}$$

determine the coefficients of $P_L(x)$ and $Q_M(x)$ by the equation. Since we can clearly multiply the numerator and denominator by a constant and leave $[L/M]$ unchanged, we imposed the normalization condition

$$Q_M(0)=1.0. \tag{9}$$

Finally, we require that $P_L(x)$ and $Q_M(x)$ have non-common factors. If we write the coefficient of $P_L(x)$ and $Q_M(x)$ as

$$\left.\begin{aligned} P_L(x) &= p_0+p_1x+p_2x^2+\cdots+p_Lx^L, \\ Q_M(x) &= q_0+q_1x+q_2x^2+\cdots+q_Mx^M \end{aligned}\right\}. \tag{10}$$

Then by (9) and (10), we may multiply (8) by $Q_M(x)$, which linearizes the coefficient equations. We can write out (8) in more details as

$$\left.\begin{array}{l} a_{L+1}+a_L q_1+\cdots+a_{L-M}q_M=0, \\ q_{L+2}+q_{L+1}q_1+\cdots+a_{L-M+2}q_M=0, \\ \vdots \\ a_{L+M}+a_{L+M-1}q_1+\cdots+a_L q_M=0, \end{array}\right\} \tag{11}$$

$$\left.\begin{array}{l} a_0=p_0, \\ a_0+a_0 q_1+\cdots+=p_1, \\ \vdots \\ a_L+a_{L-1}q_1+\cdots+a_0 q_L=p_L \end{array}\right\}. \tag{12}$$

To solve these equations, we start with equation (11), which is a set of linear equations for all the unknown q's. Once the q's are known, then equation (12) gives and explicit formula fro the unknown p's, which complete the solution. If equations (11) and (12) are nonsingular, then we can solve them directly and obtain equation (13) (see Boyd (1997), Momani (2007), where equation (13) holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$\left[\frac{L}{M}\right]=\frac{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{\det\begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \vdots & \vdots & \ddots & \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}. \tag{13}$$

To obtain diagonal Padé approximants of different order Such as [2/2], [4/4] or [6/6] we can use the symbolic calculus software, Mathematica or Maple.

4. Numerical Applications

In this section, we apply the homotopy perturbation method reviewed in section 2 for solving Flierl-Petviashivili equation. The Flierl-Petviashivili equation is converted to a first order initial value problem by introducing a new transformation which plays a fundamental and pivotal role in the application of homotopy perturbation method. The suggested homotopy perturbation method is applied to the reformulated initial value problem. Finally, the desired series solution is obtained by employing the inverse transformation. To make the work more concise, diagonal Padé approximants are used to the obtained power series.

Example 4.1.

Consider the generalized variant of the Flierl-Petviashivili equation

$$y'' + \frac{1}{x} y' - y^n - y^{n+1} = 0, \tag{14}$$

with boundary conditions

$$y(0) = \alpha, \qquad y'(0) = 0, \qquad y(\infty) = 0. \tag{15}$$

For $n = 1$, above equation reduces to the standard Flierl-Petviashivili equation. The general series solution for the equation is to be constructed for all possible values of $n \geq 1$. Using the transformation $u(x) = x\,y'(x)$, the generalized Flierl-Petviashivili equation (14, 15) can be converted to the following first order initial value problem

$$u'(x) = x\left(\int_0^x \left(\left(\frac{u(x)}{x}\right)^n + \left(\frac{u(x)}{x}\right)^{n+1}\right) dx,\right. \tag{16}$$

with initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{17}$$

The initial value problem (16) and (17) can be written as the following integral equation

$$u(x) = \int_0^x x\left(\int_0^x \left(\left(\frac{u(x)}{x}\right)^n + \left(\frac{u(x)}{x}\right)^{(n+1)} dx\right) dx.\right.$$

Applying the convex homotopy method

$$u_0(x) + pu_1(x) + p^2u_2(x) + \cdots$$
$$= p\int_0^x x\left(\begin{array}{l}\int_0^x \left(\frac{u_0 + pu_1 + p^2u_2 + \cdots}{x}\right)^n dx \\ + \int_0^x \left(\frac{u_0 + pu_1 + p^2u_2 + \cdots}{x}\right)^{n+1} dx\end{array}\right) dx.$$

Comparing the coefficients of like powers of p

$$p^{(0)}: \quad u_0(x) = 0,$$

$$p^{(1)}: \quad u_1(x) = \left(\frac{\alpha^n + \alpha^{n+1}}{2}\right) x^2,$$

$$p^{(2)}: \quad u_2(x) = \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{16\alpha} x^4,$$

$$p^{(3)}: \quad u_3(x) = \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{384\alpha^2} x^6,$$

$$p^{(4)}: \quad u_4(x) = \frac{(\alpha^n + \alpha^{n+1})(n(18n^2 - 29n + 12)\alpha^{3n} + n(54n^2 - 33n + 7)\alpha^{3n+1} + (18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}))}{18432\,\alpha^3} x^8$$

$$\vdots$$

The solution in a series form is given by

$$\begin{aligned} u(x) = {} & \frac{(\alpha^n + \alpha^{n+1})}{2} x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{16\alpha} x^4 \\ & + \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{384\alpha^2} x^6 \\ & + \frac{(\alpha^n + \alpha^{n+1})(n(18n^2 - 29n + 12)\alpha^{3n} + n(54n^2 - 33n + 7)\alpha^{3n+1} + (18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}))}{18432\alpha^3} x^8 \\ & + \cdots, \end{aligned}$$

and the inverse transformation yields:

$$\begin{aligned} y(x) = {} & \alpha + \frac{(\alpha^n + \alpha^{n+1})}{4} x^2 + \frac{(\alpha^n + \alpha^{n+1})(n\alpha^n + (n+1)\alpha^{n+1})}{64\alpha} x^4 \\ & + \frac{(\alpha^n + \alpha^{n+1})(2n(3n-1)\alpha^{2n} + 2n(3n+1)\alpha^{2n+1} + (3n+1)(n+1)\alpha^{2n+2})}{2304\,\alpha^2} x^6 \\ & + \frac{(\alpha^n + \alpha^{n+1})(n(18n^2 - 29n + 12)\alpha^{3n} + n(54n^2 - 33n + 7)\alpha^{3n+1} + (18n^2 + 7n + 1)(3n\alpha^{3n+2} + (n+1)\alpha^{3n+3}))}{147456\alpha^3} x^8 \\ & + \cdots. \end{aligned}$$

The diagonal Padé approximants can be applied to find the roots of FP equation (see Wazwaz (2006), Boyd (1997)).

Table 4.1 Roots of the Padé approximants monopole (Wazwaz (2006), Boyd (1997)) α, $n=1$

Degree	Roots
[2/2]	-1.5
[4/4]	-2.50746
[6/6]	-2.390278
[8/8]	-2.392214

Table 4.2 Roots of the Padé approximants monopole (Wazwaz (2006), Boyd (1997)) α, $n=2$

Degree	Roots
[2/2]	-2.0
[4/4]	-2.0
[6/6]	-2.0
[8/8]	-2.0

Table 4.3 Roots of the Padé approximants monopole (Wazwaz (2006), Boyd (1997)) α, $n=3$

Degree	Roots
[2/2]	0.0
[4/4]	-2.197575908
[6/6]	-1.1918424398
[8/8]	-1.848997181

Table 4.4 Roots of the Padé approximants [8/8] monopole α for several values of n

n	[8/8] Roots	n	[8/8] Roots
1	-2.392213866	7	-1.000708285
2	-2.0	8	-1.000601615
3	-1.848997181	9	-1.000523005
4	-1.286025892	10	-1.000462636
5	-1.001101141	11	-1.000262137
6	-1.000861533	$n \to \infty$	-1.0

Table 4.4 shows that the roots of the monopole α converge to -1 as n increases.

5. Conclusion

In this paper, we have shown that with the proper use of homotopy perturbation method, it is possible to attain an analytical solution of the Flierl-Petviashivili equation. The difficulty of applying the homotopy perturbation method directly is overcome here by introducing a new transformation which converts the Flierl-Petviashivili equation to a first order initial value problem and gives the solution in terms of transformed variable. The desired solution was obtained by making use of inverse transformation. The fact that the developed algorithm solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

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