




12-2008

A Reliable Approach for Higher-order Integro-differential Equations

Muhammad A. Noor
COMSATS Institute of Information Technology

Syed T. Mohyud-Din
COMSATS Institute of Information Technology

Follow this and additional works at: <https://digitalcommons.pvamu.edu/aam>

 Part of the [Numerical Analysis and Computation Commons](#), and the [Ordinary Differential Equations and Applied Dynamics Commons](#)

Recommended Citation

Noor, Muhammad A. and Mohyud-Din, Syed T. (2008). A Reliable Approach for Higher-order Integro-differential Equations, *Applications and Applied Mathematics: An International Journal (AAM)*, Vol. 3, Iss. 2, Article 3.

Available at: <https://digitalcommons.pvamu.edu/aam/vol3/iss2/3>

This Article is brought to you for free and open access by Digital Commons @PVAMU. It has been accepted for inclusion in *Applications and Applied Mathematics: An International Journal (AAM)* by an authorized editor of Digital Commons @PVAMU. For more information, please contact hvkoshy@pvamu.edu.



A Reliable Approach for Higher-order Integro-differential Equations

Muhammad Aslam Noor
Syed Tauseef Mohyud-Din

Department of Mathematics
 COMSATS Institute of Information Technology
 Islamabad, Pakistan
noormaslam@hotmail.com
syedtauseefs@hotmail.com

Received: October 25, 2007; Accepted: April 29, 2008

Abstract

In this paper, we apply the variational iteration method (VIM) for solving higher-order integro differential equations by converting the problems into system of integral equations. The proposed technique is applied to the re-formulated system of integro-differential equations. Numerical results show the accuracy and efficiency of the suggested algorithm. The fact that the VIM solves nonlinear problems without calculating Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Keywords: Boundary value problems; integro-differential equations; variational iteration method; Blasius problem; Pade' approximants.

AMS 2000 Subject Classification Numbers: 65 N 10, 34 Bxx.

1. Introduction

In this paper, we consider the general higher-order integro-differential equation of the type

$$y^{(m)}(x) = f(x) + \int_0^x K(x,t)F(y(t))dt,$$

with boundary conditions

$$y^{(j)}(0) = A_j, \quad j = 0, 1, 2, \dots, (r-1),$$

$$y^{(j)}(b) = C_j, \quad j = r, (r+1), (r+2), \dots, (m-1),$$

where $y^m(x)$ indicates the m th derivative of $y(x)$ and $F(y(x))$ is a nonlinear function. In addition the kernel $k(x, t)$, $f(x)$ are assumed real, differential for $x \in [0, b]$ and A_j , $0 \leq j \leq (r-1)$, C_j , $r \leq j \leq (m-1)$ are real finite constants. The higher-order integro-differential equations occur in multiple diversified physical phenomena and various techniques including decomposition have been applied for solving such problems, (see Agarwal (1986), Morchalo (1975a), Agarwal (1983), Morchalo (1975b), Wazwaz (2000)). He developed the variational iteration method (VIM) for solving linear and nonlinear versatile physical problems, (see He (2006), (2007)).

It is worth mentioning that the VIM was originated by Inokuti et al. (see Inokuti, Sekine and Mura (1978)). The method has been successfully applied on a wide class of initial and boundary value problems by various authors, (see Mohyud-Din, Noor and Noor (2008a), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b), Noor and Mohyud-Din (2008b), Noor and Mohyud-Din (2008c), Noor and Mohyud-Din (2008d), Noor and Mohyud-Din (2008e)). Inspired and motivated by the ongoing research in this area, we implement the variational iteration method (VIM) for solving higher-order integro-differential equations by converting the problems into systems of integral equations.

The proposed algorithm is applied on the re-formulated systems of integral equations. The application of VIM on a re-formulated system of integral equations was developed and implemented first by Noor and Mohyud-Din, (see Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b)). The method has been used without any perturbation, restrictive assumptions or discretization. Moreover, the proposed technique is free from the complexities arising in calculating Adomian's polynomials.

2. Variational iteration technique

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \quad (1)$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method, (see He (2006), (2007), Inokuti, Sekine and Mura (1978), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b)), we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s)) ds, \quad (2)$$

where λ is a Lagrange multiplier, which can be identified optimally via variational iteration method. The subscripts n denote the n th approximation, \tilde{u}_n is considered as a restricted variation. i.e. $\delta \tilde{u}_n = 0$; (2) is called as a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in He (2006), (2007), Inokuti, Sekine and Mura (1978), Noor and Mohyud-Din (2007a), Noor and Mohyud-Din (2007b), Noor and Mohyud-Din (2008f), Noor and

Mohyud-Din (2008g), Noor and Mohyud-Din (2008h)). For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$x_i'(t) = f_i(t, x_1, x_2, \dots, x_n) \quad (3)$$

$$x_i'(t) = f_i(t, x_i), \quad i = 1, 2, 3, \dots, n,$$

subject to the boundary conditions $x_i(0) = c_i, \quad i = 1, 2, 3, \dots, n$.

To solve the system by means of the variational iteration method, we rewrite the system (3) in the following form:

$$x_i'(t) = f_i(x_i) + g_i(t), \quad i = 1, 2, 3, \dots, n. \quad (4)$$

subject to the boundary conditions $x_i(0) = c_i, \quad i = 1, 2, 3, \dots, n$ and g_i is defined in (1). The correction functional for the nonlinear system (4) can be approximated as:

$$\begin{aligned} x_i^{(k+1)}(t) &= x_i^{(k)}(t) + \int_0^t \lambda_1 \left(x_1^{(k)}(T), f_1(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)) - g_1(T) \right) dT, \\ x_2^{(k+1)}(t) &= x_2^{(k)}(t) + \int_0^t \lambda_2 \left(x_2^{(k)}(T), f_2(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)) - g_2(T) \right) dT, \\ &\vdots \\ x_n^{(k+1)}(t) &= x_n^{(k)}(t) + \int_0^t \lambda_n \left(x_n^{(k)}(T), f_n(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \dots, \tilde{x}_n^{(k)}(T)) - g_n(T) \right) dT. \end{aligned} \quad (5)$$

where $\lambda_i = \pm 1, \quad i = 1, 2, 3, \dots, n$ are Lagrange multipliers, $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ denote the restricted variations.

For $\lambda_i = -1, \quad i = 1, 2, 3, \dots, n$ we have the following iterative schemes:

$$\begin{aligned} \lambda_i'(T) \Big|_{T=t} &= 0, \\ 1 + \lambda_i'(T) \Big|_{T=t} &= 0. \end{aligned}$$

For $i = 1, 2, 3, \dots, n$. Therefore the Lagrange multipliers can be easily identified as:

$$\lambda_i = \pm 1, \quad i = 1, 2, 3, \dots, n. \quad (6)$$

Substituting (6) into the correct functional (5), results the following iteration formula:

$$\begin{aligned}
x_1^{(k+1)}(t) &= x_1^{(k)}(t) - \int_0^t \left(x_1^{(k)}(T), f_1(x_1^{(k)}(T), x_2^{(k)}(T), \dots, x_n^{(k)}(T)) - g_1(T) \right) dT, \\
x_2^{(k+1)}(t) &= x_2^{(k)}(t) - \int_0^t \left(x_2^{(k)}(T), f_2(x_1^{(k)}(T), x_2^{(k)}(T), \dots, x_n^{(k)}(T)) - g_2(T) \right) dT, \\
&\vdots \\
x_n^{(k+1)}(t) &= x_n^{(k)}(t) - \int_0^t \left(x_n^{(k)}(T), f_n(x_1^{(k)}(T), x_2^{(k)}(T), \dots, x_n^{(k)}(T)) - g_n(T) \right) dT.
\end{aligned} \tag{7}$$

If we start with the initial approximations $x_i(0) = c_i$, $i = 1, 2, 3, \dots, n$, then the approximations can be completely determined; finally we approximate the solution $x_i(t) = \lim_{k \rightarrow \infty} x_i^{(k)}(t)$ by the n^{th} term $x_i^{(n)}(t)$ for $i = 1, 2, 3, \dots, n$.

3. Numerical Applications

In this section, we first show that the higher order integro differential equations can be re-written in the form of a system of integral equations by using a suitable transformation. The VIM is used for solving the re-formulated system of integral equations. For the sake of comparison, we take the same examples as discussed in Wazwaz (2000).

Example 3.1.

Consider the linear boundary value problem for the fourth-order integro differential equation:

$$y^{(iv)}(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t) dt, \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 1, \quad y'(0) = 1, \quad y(1) = 1 + e, \quad y'(1) = 2e.$$

Using the transformation $\frac{dy}{dx} = q(x)$, $\frac{dq}{dx} = f(x)$, $\frac{dt}{dx} = z(x)$, the above boundary value problem

can be transformed as:

$$\begin{cases} \frac{dy}{dx} = q(x), & \frac{dq}{dx} = f(x), \\ \frac{dt}{dx} = z(x), & \frac{dz}{dx} = x(1 + e^x) + 3e^x + y(x) + \int_0^x y(t) dt, \end{cases}$$

with boundary conditions

$$y(0) = 1, \quad q(0) = 1, \quad f(0) = A, \quad z(0) = B.$$

The exact solution of the above boundary value problem is $y(x) = 1 + x e^x$.

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = +1, i = 1, 2, 3, 4$.

$$\begin{cases} y^{(k+1)}(x) = 1 + \int_0^x q^{(k)}(t)dt, & q^{(k+1)}(x) = -1 + \int_0^x f^{(k)}(t)dt, \\ f^{(k+1)}(x) = A + \int_0^x z^{(k)}(t)dt, & z^{(k+1)}(x) = B + \int_0^x \left((t(1 + e^t) + 3e^t + y^{(k)}(x))dx + \int_0^x y^{(k)}(t) \right) dt, \end{cases}$$

where

$$A = y''(0), \quad B = y'''(0).$$

Consequently, following approximants are obtained

$$y^{(0)}(x) = 1, q^{(0)}(x) = 1, f^{(0)}(x) = A, z^{(0)}(x) = B,$$

$$\begin{cases} y^{(1)}(x) = 1 + x, & q^{(1)}(x) = 1 + Ax, \\ f^{(1)}(x) = A + Bx, & z^{(1)}(x) = B - 2 + x + x^2 + 2e^x + xe^x, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = 1 + x + \frac{1}{2} Ax^2, & q^{(2)}(x) = 1 + Ax + \frac{1}{2} Bx^2, \\ f^{(2)}(x) = A + Bx - 1 - 2x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + e^x + xe^x, & z^{(2)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!} x^2 - \frac{1}{3!} x^3 \end{cases}$$

$$\begin{cases} y^{(3)}(x) = 1 + x + \frac{1}{2} Ax^2 + \frac{3}{3!} Bx^3, & q^{(3)}(x) = 1 + Ax + \frac{1}{2} Bx^2 - x - x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + xe^x, \\ f^{(3)}(x) = A + Bx - 1 - 2x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + e^x + xe^x - \frac{1}{3!} x^3 - \frac{1}{4!} x^4, \\ z^{(3)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{3!} Ax^3 + \frac{1}{4!} Ax^4, \end{cases}$$

$$\left\{ \begin{array}{l} y^{(4)}(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{3}{3!}Bx^3 + 1 - \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - e^x + xe^x, \\ q^{(4)}(x) = 1 + Ax + \frac{1}{2}Bx^2 - x - x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + xe^x - \frac{1}{4!}x^4 - \frac{1}{5!}x^5, \\ f^{(4)}(x) = A + Bx - 1 - 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^x + xe^x - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}Ax^4 + \frac{1}{5!}Ax^5, \\ z^{(4)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{3!}Ax^3 + \frac{1}{4!}Ax^4 + \frac{3}{4!}Bx^4 + \frac{3}{5!}Bx^5, \\ y^{(5)}(x) = 1 + x + \frac{1}{2}Ax^2 + \frac{3}{3!}Bx^3 + 1 - \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - e^x + xe^x - \frac{1}{5!}x^5 - \frac{1}{6!}x^6, \\ q^{(5)}(x) = 1 + Ax + \frac{1}{2}Bx^2 - x - x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + xe^x - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6, \\ f^{(5)}(x) = A + Bx - 1 - 2x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + e^x + xe^x - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}Ax^4 + \frac{1}{5!}Ax^5 + \frac{3}{5!}Bx^5 + \frac{3}{6!}Bx^6, \\ z^{(5)}(x) = B - 2 + x + x^2 + 2e^x + xe^x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{3!}Ax^3 + \frac{1}{4!}Ax^4 + \frac{3}{4!}Bx^4 + \frac{3}{5!}Bx^5 + 5 \\ \quad + 3x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{2}{4!}x^4 + \frac{2}{6!}x^6 + \frac{1}{7!}x^7 - 5e^x + xe^x, \\ \vdots \end{array} \right.$$

The series solution is given as

$$\begin{aligned} y(x) = & 1 + x + \frac{1}{2!}Ax^2 + \frac{1}{6}Bx^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \left(\frac{1}{720}A + \frac{1}{180}\right)x^6 + \left(\frac{1}{840} - \frac{1}{5040}A + \frac{1}{5040}B\right)x^7 \\ & + \left(\frac{11}{40320} - \frac{1}{40320}B\right)x^8 + \frac{1}{40320}x^9 + \left(\frac{1}{453600} + \frac{1}{3628800}A\right)x^{10} \\ & + \left(-\frac{1}{19958400}A + \frac{1}{39916800}B + \frac{1}{3326400}\right)x^{11} + \left(\frac{1}{479001600}A - \frac{1}{239500800}B + \frac{1}{29937600}\right)x^{12} + \dots \end{aligned}$$

Imposing the boundary conditions at $x = 1$, we obtained

$$A = 1.999999953, \quad B = 3.000000151.$$

$$\begin{aligned} y(x) = & 1 + x + 0.9999999765x^2 + 0.5000000252x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + 0.008333333269x^6 + 0.001388888928x^7 \\ & + 0.0001984126946x^8 + \frac{1}{40320}x^9 + 0.27557310 \times 10^{-5}x^{10} + 0.2755731983 \times 10^{-6}x^{11} \\ & + 0.2505210766 \times 10^{-7}x^{12} + \dots \end{aligned}$$

Table 3.1 (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.11105170920	1.1105170920	2.0 E-10
0.2	1.2442805520	1.1.2442805510	6.09 E-10
0.3	1.4049576420	1.4049576410	1.4 E-9
0.4	1.5967298790	1.5967298780	1.2 E-9
0.5	1.8243606360	1.8243606320	3.5 E-9
0.6	2.0932712800	2.0932712780	2.0 E-9
0.7	2.4096268950	2.4096268920	3.0 E-9
0.8	2.7804327420	2.7804327410	1.0 E-9
0.9	3.2136428000	3.2136427980	2.0E-9
1.0	3.7182818280	3.7182818290	1.0E-9

*Error=Exact solution-Series solution.

Table 3.1 exhibits the errors obtained by applying the variational iteration method. Higher accuracy can be obtained by using some more terms of the series solution.

Example 3.2.

Consider the nonlinear boundary value problem for the integro differential equation

$$y^{(iv)}(x) = 1 + \int_0^x e^{-x} y^2 dx, \quad 0 < x < 1$$

with boundary conditions

$$y(0) = 1, \quad y'(0) = 1, \quad y(1) = e, \quad y'(1) = e.$$

The exact solution of the above boundary value problem is $y(x) = e^x$.

Using transformation $\frac{dy}{dx} = q(x)$, $\frac{dq}{dx} = f(x)$, $\frac{df}{dx} = z(x)$, the above boundary value problems

can be transformed as the following system of differential equations

$$\begin{cases} \frac{dy}{dx} = q(x), & \frac{dq}{dx} = f(x), \\ \frac{df}{dx} = z(x), & \frac{dz}{dx} = 1 + \int_0^x e^{-x} y^2(x) dx, \end{cases}$$

with boundary conditions

$$y(0) = 1, \quad q(0) = 1, \quad f(0) = A, \quad z(0) = B.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1$, $i = 1, 2, 3, 4$.

$$\begin{cases} y^{(k+1)}(x) = 1 + \int_0^x q(t) dt, & q^{(k+1)}(x) = 1 + \int_{0,x} f(t) dt, \\ f^{(k+1)}(x) = A + \int_0^x z(t) dt, & z^{(k+1)}(x) = B + \int_0^x \left(1 + \int_0^t e^{-s} (y^{(k)}(s))^2 ds \right) dt. \end{cases}$$

Consequently, following approximants are obtained:

$$y^{(0)}(x) = 1, \quad q^{(0)}(x) = 1, \quad f^{(0)}(x) = A, \quad z^{(0)}(x) = B,$$

$$\begin{cases} y^{(1)}(x) = 1 + x, & q^{(1)}(x) = 1 + Ax, \\ f^{(1)}(x) = A + Bx, & z^{(1)}(x) = B + 4 - 4e^{-x}, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = 1 + x + \frac{1}{2!} Ax^2, & q^{(2)}(x) = 1 + Ax + \frac{1}{3!} Bx^2, \\ f^{(2)}(x) = A + Bx - 4 + 4x + 4e^{-x}, & z^{(2)}(x) = B + 8 - 8e^{-x} - 4xe^{-x}, \end{cases}$$

$$\begin{cases} y^{(3)}(x) = 1 + x + \frac{1}{2!} Ax^2 + \frac{3}{4!} Bx^3, & q^{(3)}(x) = q^{(2)}(x) = 1 + Ax + \frac{1}{3!} Bx^2 + 4 - 4x + \frac{4}{2!} - 4e^{-x}, \\ f^{(3)}(x) = A + (B + 8)x + 4e^{-x} - 4xe^{-x}, \\ z^{(3)}(x) = z^{(2)}(x) = B + 8 - 8e^{-x} - 4xe^{-x} + \frac{5}{2!} A(2 - 2e^{-x} - 2xe^{-x} - x^2e^{-x}), \end{cases}$$

⋮

The series solution is given by:

$$\begin{aligned} y(x) = & 1 + x + \frac{1}{2!} Ax^2 + \frac{1}{3!} Bx^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \frac{1}{6!} x^6 + \left(\frac{1}{2520} A - \frac{1}{1680} \right) x^7 + \left(-\frac{1}{6720} A + \frac{1}{20160} B + \frac{1}{8064} \right) x^8 \\ & + \left(\frac{1}{30240} A - \frac{1}{45360} B + \frac{1}{72576} \right) x^9 + \left(-\frac{1}{181440} A + \frac{1}{181440} B + \frac{1}{10!} \right) x^{10} \\ & + \left(\frac{1}{1330560} A - \frac{1}{997920} B + \frac{1}{7983360} \right) x^{11} + \left(-\frac{1}{11404800} A + \frac{1}{6842880} B + \frac{1}{12!} \right) x^{12} + \dots \end{aligned}$$

Imposing the boundary conditions at $x = 1$, we obtained

$$A = 0.9970859583, \quad B = 1.010994057.$$

The series solution is given as

$$y(x) = 1 + x + 0.4985429792x^2 + 0.1684990095x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + -0.0001995690641x^7 \\ + 0.00002578056453x^8 + 0.00002446285010x^9 + 0.3522271752 \times 10^{-6}x^{10} \\ - 0.1384676012 \times 10^{-6}x^{11} + 0.624047474716 \times 10^{-7}x^{12} + \dots$$

Table 3.2. (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.1051581800	1.1051581800	1.27 E-5
0.2	1.2214027580	1.2213591310	4.36 E-5
0.3	1.3498588080	1.3497770620	8.17 E-5
0.4	1.4918246980	1.4917081990	1.16 E-4
0.5	1.6487212710	1.6485829960	1.38 E-4
0.6	1.8221188000	1.8219791520	1.39 E-4
0.7	2.0137527070	2.0136354170	1.17-4
0.8	2.2255409280	2.2254662080	7.47 E-5
0.9	2.4596031110	2.4595771740	2.59 E-5
1.0	2.7182818280	2.7182818280	0.000000

*Error=Exact solution-Series solution.

Table 3.2 exhibits the errors obtained by applying the variational iteration method. Higher accuracy can be obtained by using some more terms of the series solution.

Example 3.3.

Consider the two dimensional nonlinear inhomogeneous boundary value problem for the integro differential equation related to the Blasius problem

$$y''(x) = \alpha - \frac{1}{2} \int_0^x y(t)y''(t)dt, \quad -\infty < x < 0$$

with boundary conditions

$$y(0) = 0, \quad y'(0) = 1.$$

and

$$\lim_{x \rightarrow -\infty} y'(x) = 0,$$

where α is a constant and is given by

$$y''(0) = \alpha, \quad \alpha > 0.$$

Using the transformation $\frac{dy}{dx} = q(x)$, the above boundary value problem can be written as the following system of differential equations

$$\begin{cases} \frac{dy}{dx} = q(x), \\ \frac{dq}{dx} = \alpha - \frac{1}{2} \int_0^x y(t)y''(t)dt, \end{cases}$$

with boundary conditions

$$y(0) = 0, \quad q(0) = 1, \quad q'(0) = \alpha.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = 1, i = 1, 2$.

$$\begin{cases} y(x) = \int_0^x q^{(k)}(t)dt, \\ q(x) = 1 + \int_0^x \left(\alpha - \frac{1}{2} \int_0^x y^{(k)}(t)q'^{(k)}(t)dt \right) dt. \end{cases}$$

Consequently, the following approximants are obtained

$$y^{(0)}(x) = 1, \quad q^{(0)}(x) = 1,$$

$$\begin{cases} y^{(1)}(x) = x, \\ q^{(1)}(x) = 1 + \alpha x, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = x + \frac{1}{2} \alpha x^2, \\ q^{(2)}(x) = 1 + \alpha x - \frac{1}{12} \alpha x^3, \end{cases}$$

$$\begin{cases} y^{(3)}(x) = x + \frac{1}{2} \alpha x^2 - \frac{1}{48} \alpha x^4, \\ q^{(3)}(x) = 1 + \alpha x - \frac{1}{12} \alpha x^3 - \frac{1}{12} \alpha x^3 - \frac{1}{48} \alpha x^4 + \frac{1}{160} \alpha x^5 + \frac{1}{480} \alpha^2 x^6, \end{cases}$$

\vdots

The series solution is given as:

$$y(x) = x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \frac{1}{960}\alpha x^6 + \frac{11}{20160}\alpha^2 x^7 + \left(\frac{11}{161280}\alpha^3 + \frac{1}{960}\alpha\right)x^8 - \frac{43}{967680}\alpha^2 x^9 + \left(\frac{1}{52960}\alpha - \frac{5}{387072}\alpha^3\right)x^{10} + \left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{11} + \left(-\frac{1}{16220160}\alpha + \frac{1}{7257792}\alpha^3\right)x^{12} + \dots,$$

and consequently

$$y'(x) = 1 + \alpha x - \frac{1}{12}\alpha x^3 - \frac{1}{48}\alpha^2 x^4 + \frac{1}{160}\alpha x^5 + \frac{11}{2880}\alpha^2 x^6 \left(\frac{11}{20160}\alpha^3 - \frac{1}{2688}\alpha\right)x^7 - \frac{43}{107520}\alpha^2 x^8 + 10\left(\frac{1}{552960}\alpha - \frac{5}{387072}\alpha^3\right)x^9 + 11\left(\frac{587}{212889600}\alpha^2 - \frac{5}{4257792}\alpha^4\right)x^{10} + 12\left(-\frac{1}{16220160}\alpha + \frac{1}{725760}\alpha^3\right)x^{11} + \dots,$$

is obtained. The diagonal Pade' approximants can be applied, consequently, we obtained values of the constant α which are listed in the table 3.3, (see Wazwaz (2000)).

Table 3.3

Pade' approximants	α
[2/2]	0.5778502691
[3/3]	0.5163977793
[4/4]	0.5227030798

Table 3.3 Pade' approximants and numerical value of α .

The above results are in full agreement with the calculations in Wazwaz (2000), (1997).

4. Conclusion

In this paper, we applied the variational iteration method (VIM) for solving the higher order integro differential equations by using the re-formulated system of integral equations. The method is used in a direct way without using linearization, perturbation, discretization or restrictive assumption. The method gives more realistic series solutions that converge very rapidly in physical problems. Thus we conclude that the variational iteration method can be considered as an efficient and effective method for solving linear and nonlinear initial and boundary value problems. The fact that the VIM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

Acknowledgement:

The authors are highly grateful to both the referees and Prof Dr A. M. Haghghi for their constructive comments. We would like to thank Dr. S. M. Junaid Zaidi, Rector CIIT for the provision of excellent research environment and facilities.

REFERENCES

- Agarwal, R. P. (1986). Boundary value problems for higher-order differential equations, World Scientific, Singapore.
- Morchalo, J. (1975a). On two point boundary value problem for integro-differential equation of second order Fasc, Math, 9, pp. 51-56.
- Morchalo, J. (1975b). On two point boundary value problem of integro-differential equation of higher order, Fasc. Math. 9, pp. 77-96.
- Agarwal, R. P. (1983). Boundary value problems for higher order integro-differential equations Nonlinear Analysis, Theory, Math. Appl. 7 (3), pp. 259-270.
- He, J. H. (2006). Some asymptotic methods for strongly nonlinear equation, Int. J. Mod. Phys. 20 (20), pp.1144-1199.
- He, J. H. (2007). Variational iteration method—some recent results and new interpretations, J. Comput. Appl. Math. 207, pp. 3-17
- Inokuti, M., Sekine, H., and Mura, T. (1978). General use of the Lagrange multiplier in nonlinear mathematical physics, in: Variational Method in the Mech. of Solids, Pergamon Press, New York, pp. 156-162.
- Mohyud-Din, S. T., Noor, M.A. and Noor, K. I. (2008a). Travelling wave solutions of seventh-order generalized KdV equations by variational iteration method using Adomian's polynomials, Int. J. Mod. Phys. B.
- Noor, M. A. and Mohyud-Din, S. T. (2007a). Variational iteration technique for solving higher order boundary value problems, Appl. Math. Comput. 189, pp. 1929-1942.
- Noor, M. A. and Mohyud-Din, S. T. (2007b). An efficient method for fourth order boundary value problems, Comput. Math. Appl. 54, pp. 1101-1111.
- Noor, M. A. and Mohyud-Din, S. T. (2008b). Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, Int. J. Nonlin. Sci. Num. Simul. 9 (2), pp. 141-157.
- Noor, M. A. and Mohyud-Din, S. T. (2008c). Variational iteration method for solving fifth-order boundary value problems using He's polynomials, Math. Prob. Engg. 2008.
- Noor, M. A. and Mohyud-Din, S. T. (2008d). Modified variational iteration method for heat and wave-like equations, Acta Applnda. Mathmtce.
- Noor, M. A. and Mohyud-Din, S. T. (2008e). Modified variational iteration method for solving fourth-order boundary value problems, J. Appl. Math. Computg.
- Noor, M. A. and Mohyud-Din, S. T. (2008f). Variational homotopy perturbation method for solving higher dimensional initial boundary value problems, Math. Prob. Engg. 2008.
- Noor, M. A. and Mohyud-Din, S. T. (2008g). Variational iteration method for unsteady flow of gas through a porous medium using He's polynomials and Pade' approximants, Comput. Math. Appl.
- Noor, M. A. and Mohyud-Din, S. T. (2008h). Solution of singular and nonsingular initial and boundary value problems by modified variational iteration method, Math. Prob. Engg. 2008.
- Wazwaz, A. M. (2000). A reliable algorithm for solving boundary value problems for higher order integro differential equations, Appl. Math. Comput. 118, pp. 327-342.