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Abstract

The aim of the present paper is to study the distribution of the mixed sum of two random variables. Here we establish a theorem which gives the probability density function (pdf) of sum of doubly infinite and finite independent random variables. The distribution of the infinite and finite independent random variables is given in the form of corollary. As an application of these results we have obtained a distribution of sum of bilateral exponential variate with triangular, Rayleigh with uniform and Weibull with triangular variate. Some graphs of these distributions have also been given.

Keywords: Doubly infinite distribution, exponential bilateral distribution, finite distribution, mixed sum, triangular distribution.

MSC(2000): 62E99, 60E05, 60E99.

1. Introduction

Since 1934, the problem of deriving the distribution of sums of random variables has received a great deal of attention and systematic procedures for determining such distributions have been well developed by many authors such as Aroian (1944), Witner (1934), Lukacs (1970, 1964) and Cramer (1951, 1962) to mention a few. The distribution of sum of random variables is of interest in many areas of physics and engineering. For example, sums of independent gamma random variables have application in problems of queuing theory such as determination of total waiting time, in civil engineering such as determination of the total excess water flow in a dam. They also appear in obtaining the inter arrival time of drought events which is the sum of the drought duration and the successive non drought duration. In the works of Linhart (1965) and Jackson (1969) they appear in test statistics used to determine the confidence limits for the coefficient of variation of fiber diameters. Grice and Bain (1980) have studied it in connection with the inference about the mean of the two parameter gamma distribution.

The distribution of sum of two independent random variables has been obtained by several authors, particularly when both the variates come from the same family of distributions. In this context the works of Albert (2002) for uniform variates, Moschopoulos (1985), Provost (1979) and Holmes (2004) for gamma variates, Van Dorp and Kotz (2003) for triangular variates sharing the same support (but not necessarily the same mode) are worth mentioning. In a paper by Loaiciga and Leipnik(1999) the probability distribution of sum of two Gumble variables has been derived and the authors have given several examples of its application in hydrology. Agrawal and Elmaghraby (2001) provided computational algorithm for the distribution function of sum of independent random variables.

The literature on the sum of random variables is quite rich, however relatively little work is done when both the variate belong to different families of distributions. Recently Nason (2006) has obtained the distribution of sum of t and Gaussian random variables and pointed out its application in Bayesian wavelet shrinkage. In the present paper, we consider more general problem of obtaining the distribution of sum of two independent random variables belonging to doubly infinite and finite family of distributions respectively. The results are organized as follows. In section 2 we obtain a general theorem which gives pdf of the sum of two independent random variables belonging to doubly infinite and finite family of distributions respectively. The pdf of sum of two independent random variables belonging to infinite and finite families of distributions is derived as a corollary of the theorem. In section 3 we obtain some particular distributions as an application of the main theorem and its corollary and also plot graphs for these results.

2. Distribution of mixed sum of two independent random variables

Theorem: Let the independent random variables X_1 and X_2 have the probability density functions $f(x_1)$ and $g(x_2)$ defined as follows:

$$f(x_1) = \begin{cases} f^{+}(x_1), & x_1 \geq 0 \\ f^{-}(x_1), & x_1 < 0 \end{cases} \tag{1}$$

and

$$g(x_2) = \begin{cases} g'(x_2), & a \leq x_2 < m \\ g''(x_2), & m \leq x_2 \leq b \\ 0, & elsewhere, \end{cases} \tag{2}$$

then, the pdf $h(w)$ of $W = X_1 + X_2$ is given by

$$h(w)=\begin{cases} \int_{w-b}^{w-m} f^-(x_1)g''(w-x_1)dx_1 + \int_{w-m}^{w-a} f^-(x_1)g'(w-x_1)dx_1, & (-\infty < w \le a) \\ \int_{w-b}^{w-m} f^-(x_1)g''(w-x_1)dx_1 + \int_{w-m}^{0} f^-(x_1)g'(w-x_1)dx_1 + \int_{0}^{w-a} f^+(x_1)g'(w-x_1)dx_1, & (a < w \le m) \\ \int_{w-m}^{w-a} f^+(x_1)g'(w-x_1)dx_1 + \int_{w-b}^{0} f^-(x_1)g''(w-x_1)dx_1 + \int_{0}^{w-m} f^+(x_1)g''(w-x_1)dx_1, & (m < w \le b) \\ \int_{w-b}^{w-m} f^+(x_1)g''(w-x_1)dx_1 + \int_{w-m}^{w-a} f^+(x_1)g'(w-x_1)dx_1, & (b < w < \infty). \end{cases} \quad (3)$$

Proof: We know that pdf $h(w)$ of $W = X_1 + X_2$ is given by

$$h(w) = \int_{range\, of\, x_1} f(x_1)\, g(w-x_1)\, dx_1 . \quad (4)$$

Now $X_2 = W - X_1 \ge a \Rightarrow X_1 \le W - a$ and $X_2 = W - X_1 \le b \Rightarrow X_1 \ge W - b$, hence (4) can be written as

$$h(w) = \int_{w-b}^{w-a} f(x_1)\, g(w-x_1)\, dx_1 . \quad (5)$$

As f and g are not same throughout the interval (w-b, w-a), we shall consider the integral (5) for different ranges of values of w as follows

(i) $-\infty < w \le a$ (ii) $a < w \le m$ (iii) $m < w \le b$ (iv) $b < w < \infty$.

Considering integral (5) for the values of w given by case (i), we find that $f(x_1)$ is always negative in the range of integral whereas $g(w-x_1)$ takes values in the range $[a, b]$ where it is defined by equation(2). Hence, the integral (5) can be written as sum of two integrals as follows:

$$h(w) = \int_{w-b}^{w-m} f^-(x_1)g''(w-x_1)dx_1 + \int_{w-m}^{w-a} f^-(x_1)g'(w-x_1)dx_1 , (-\infty < w \le a). \quad (6)$$

Next, we consider the integral (5) for the values of w given by case (ii). We first write it in two parts in accordance with the definition of g in the range $[a, b]$ as

$$h(w) = \underbrace{\int_{w-b}^{w-m} f(x_1)g''(w-x_1)dx_1}_{I_1} + \underbrace{\int_{w-m}^{w-a} f(x_1)g'(w-x_1)dx_1}_{I_2} ; (a < w \le m) . \quad (7)$$

In I_1, f is always negative, i.e., $f = f^-$ whereas in I_2, f takes both positive and negative values and, hence, I_2 can be written as

$$I_2 = \int_{w-m}^{0} f^-(x_1)\, g'(w-x_1)\, dx_1 + \int_{0}^{w-a} f^+(x_1)\, g'(w-x_1)\, dx_1 . \tag{8}$$

Thus, from (7) and (8) we get the value of $h(w)$ for $a < w \le m$ as written in the theorem.

Now, we consider the integral (5) for the value of w given by case (iii). As in the above case we first write the integral (5) in two parts in accordance with the definition of g as

$$h(w) = \underbrace{\int_{w-b}^{w-m} f(x_1)\, g''(w-x_1)\, dx_1}_{I_3} + \underbrace{\int_{w-m}^{w-a} f(x_1)\, g'(w-x_1)\, dx_1}_{I_4} ;\ (m < w \le b). \tag{9}$$

In I_3, f is taking both positive and negative values whereas in I_4, f is always positive, i.e., $f = f^+$, therefore

$$I_3 = \int_{w-b}^{0} f^-(x_1)\, g''(w-x_1)\, dx_1 + \int_{0}^{w-m} f^+(x_1)\, g''(w-x_1)\, dx_1 . \tag{10}$$

Thus, from (9) and (10) we get the value of $h(w)$ for $m < w \le b$.

Finally, we consider the integral (5) for the values of w as given by case (iv), we find that $f(x_1)$ is always positive in the range of integral, whereas $g(w\text{-}x_1)$ takes values according to the definition (2). Hence, the integral can be written as sum of two integrals as follows:

$$h(w) = \int_{w-b}^{w-m} f^+(x_1)\, g''(w-x_1)\, dx_1 + \int_{w-m}^{w-a} f^+(x_1)\, g'(w-x_1)\, dx_1,\ (b < w < \infty). \tag{11}$$

Hence, the theorem.

In the above theorem if we take $f^-(x_1)=0$, we arrive at the following result giving pdf of sum of one infinite and one finite random variables.

Corollary: Let X_1 be an infinite random variable with pdf $f(x_1)$ given by

$$f(x_1) = \begin{cases} f^+(x_1), & x_1 \ge 0 \\ 0, & elsewhere \end{cases} \tag{12}$$

and X_2 be a finite random variable with pdf $g(x_2)$ as given by equation (2), then the pdf $h(w)$ of $W= X_1 + X_2$ is given as follows:

$$h(w)=\begin{cases} \int_0^{w-a} f^+(x_1)\, g'(w-x_1)\, dx_1, & (a \le w < m) \\ \int_0^{w-m} f^+(x_1)\, g''(w-x_1)\, dx_1 + \int_{w-m}^{w-a} f^+(x_1)\, g'(w-x_1)\, dx_1, & (m \le w < b) \\ \int_{w-b}^{w-m} f^+(x_1)\, g''(w-x_1)\, dx_1 + \int_{w-m}^{w-a} f^+(x_1)\, g'(w-x_1)\, dx_1, & (b \le w < \infty). \end{cases} \quad (13)$$

The definition of the finite pdf taken in the theorem includes not only the well known distributions such as Uniform, beta given in Mathai (1993), Kumaraswamy defined by Kumaraswami (1980), beta trigonometric defined by Nadarajah and Kotz (2006), beta Bessel given by Gupta and Nadarajah (2006) but the triangular distribution given in Kotz and Van Dorp (2004) as well .The triangular distribution is a very versatile distribution for modeling processes where the relationship between variables is known but data is scarce. Johnson (1997) and Kotz & Van Dorp (2004) noted the advantages of using the triangular distribution over the less user friendly beta distribution. Recently this distribution is gaining importance due to its use in discrete system simulation (2001), Monte Carlo simulation technique (2002), standard uncertainty analysis software-such as Risk or Crystal ball and in discrete event simulation software- such as Arena.

3. Applications

As an application of the theorem and corollary, we now obtain the distributions of the sum of random variables X_1 and X_2 in the following situations.

Result 1: Let the random variable X_1 follow the bilateral exponential distribution given in Springer (1979)

$$f(x_1) = e^{-|x_1|} = \begin{cases} e^{-x_1}, & x_1 \ge 0 \\ e^{x_1}, & x_1 < 0 \end{cases} \quad (14)$$

and the random variable X_2 follow the triangular distribution given in Kotz and Van Dorp (2004)

$$g(x_2) = \begin{cases} \dfrac{2(x_2 - a)}{(b-a)(m-a)}, & a \le x_2 < m \\ \dfrac{2(b - x_2)}{(b-a)(b-m)}, & m \le x_2 \le b, \end{cases} \quad (15)$$

Then, the random variable $h(w)$ of $W = X_1 + X_2$ is given by

$$h(w)=\begin{cases}\frac{1}{(b-a)(b-m)}\left\{(b-m-1)e^{(w-m)}+e^{(w-b)}\right\}+\frac{1}{(b-a)(m-a)}\left\{(a-m-1)e^{(w-m)}+e^{(w-a)}\right\}, & (-\infty<w\le a)\\ \frac{1}{(b-a)(b-m)}\left\{(b-m-1)e^{(w-m)}+e^{(w-b)}\right\}+\frac{1}{(b-a)(m-a)}\left\{(a-m-1)e^{(w-m)}+e^{-(w-a)}+2(w-a)\right\}, & (a<w\le m)\\ \frac{1}{(b-a)(b-m)}\left\{2(b-m)+(m-b-1)e^{-(w-m)}+e^{(w-b)}\right\}+\frac{1}{(b-a)(m-a)}\left\{(m-a-1)e^{-(w-m)}+e^{-(w-a)}\right\}, & (m<w\le b)\\ \frac{1}{(b-a)(b-m)}\left\{(m-b-1)e^{-(w-m)}+e^{-(w-b)}\right\}+\frac{1}{(b-a)(m-a)}\left\{(m-a-1)e^{-(w-m)}+e^{-(w-a)}\right\}, & (b<w<\infty)\end{cases} \tag{16}$$

Proof: Substituting the values of $f(x_1)$ and $g(x_2)$ from (14) and (15) in the theorem, and simplifying the integrals thus obtained we arrive at the required result (16). Figure 1 illustrates possible shape of the random variable (16) for a = -1, m = 0, b = 1.

Result 2: Let the random variable X_1 follow the Rayleigh distribution

$$f(x_1)=\frac{\theta}{2}|x_1|exp\left\{-\frac{\theta}{2}x_1^2\right\}=\begin{cases}\frac{\theta}{2}x_1\,exp\left\{-\frac{\theta}{2}x_1^2\right\}, & x_1\ge 0\\ -\frac{\theta}{2}x_1\,exp\left\{-\frac{\theta}{2}x_1^2\right\}, & x_1<0,\end{cases} \tag{17}$$

where $\theta > 0$ and the random variable X_2 follow the uniform distribution

$$g(x_2)=\begin{cases}\frac{1}{(b-a)}, & a\le x_2\le b\\ 0, & elsewhere.\end{cases} \tag{18}$$

Then, the pdf $h(w)$ of $W = X_1 + X_2$ is given by

$$h(w)=\begin{cases}\frac{1}{2(b-a)}\left[\exp\left\{-\frac{\theta}{2}(w-a)^2\right\}-\exp\left\{-\frac{\theta}{2}(w-b)^2\right\}\right], & (-\infty<w\le a)\\ \frac{1}{2(b-a)}\left[2-\exp\left\{-\frac{\theta}{2}(w-b)^2\right\}-\exp\left\{-\frac{\theta}{2}(w-a)^2\right\}\right], & (a<w<b)\\ -\frac{1}{2(b-a)}\left[\exp\left\{-\frac{\theta}{2}(w-a)^2\right\}-\exp\left\{-\frac{\theta}{2}(w-b)^2\right\}\right], & (b\le w<\infty).\end{cases} \tag{19}$$

Proof: Substituting the values of $f(x_1)$ and $g(x_2)$ from (17) and (18) in the theorem and evaluating the integrals thus obtained we arrive at the above result (19). Figure 2 illustrates possible shape of the pdf (19) for θ = 5, a = 0, b = 5.

Result 3: Let the random variable X_1 follow the Weibull distribution defined below

$$f(x_1) = \begin{cases} \dfrac{\alpha}{\theta} x_1^{\alpha-1} exp\left\{-\dfrac{1}{\theta} x_1^{\alpha}\right\}, & x_1 > 0 \\ 0, & x_1 \le 0, \end{cases} \tag{20}$$

where $\alpha > 0$, $\theta > 0$, and the random variable X_2 follow the triangular distribution as given by (15), then the pdf $h(w)$ of $W = X_1 + X_2$ is given by

$h(w) =$

$$\frac{2}{\theta(b-a)(m-a)}\left[\theta(w-a)\left(1-\exp\left\{-\frac{(w-a)^{\alpha}}{\theta}\right\}\right)-\theta^{\frac{1}{\alpha}+1}\gamma\left(\frac{1}{\alpha}+1,\frac{(w-a)^{\alpha}}{\theta}\right)\right], \ (a \le w < m)$$

$$\frac{2}{\theta(b-a)(b-m)}\left[-\theta(b-w)\left(\exp\left\{-\frac{(w-m)^{\alpha}}{\theta}\right\}-1\right)+\theta^{\frac{1}{\alpha}+1}\gamma\left(\frac{1}{\alpha}+1,\frac{(w-m)^{\alpha}}{\theta}\right)\right]$$
$$+\frac{2}{\theta(b-a)(m-a)}\left[\begin{array}{l}-\theta(w-a)\left(\exp\left\{-\frac{(w-a)^{\alpha}}{\theta}\right\}-\exp\left\{-\frac{(w-m)^{\alpha}}{\theta}\right\}\right)\\ \qquad -\theta^{\frac{1}{\alpha}+1}\left\{\gamma\left(\frac{1}{\alpha}+1,\frac{(w-a)^{\alpha}}{\theta}\right)-\gamma\left(\frac{1}{\alpha}+1,\frac{(w-m)^{\alpha}}{\theta}\right)\right\}\end{array}\right], \ (m \le w < b)$$

$$\frac{2}{\theta(b-a)(b-m)}\left[\begin{array}{l}-\theta(b-w)\left(\exp\left\{-\frac{(w-m)^{\alpha}}{\theta}\right\}-\exp\left\{-\frac{(w-b)^{\alpha}}{\theta}\right\}\right)\\ \qquad +\theta^{\frac{1}{\alpha}+1}\left\{\gamma\left(\frac{1}{\alpha}+1,\frac{(w-m)^{\alpha}}{\theta}\right)-\gamma\left(\frac{1}{\alpha}+1,\frac{(w-b)^{\alpha}}{\theta}\right)\right\}\end{array}\right]+$$
$$+\frac{2}{\theta(b-a)(m-a)}\left[\begin{array}{l}-\theta(w-a)\left(\exp\left\{-\frac{(w-a)^{\alpha}}{\theta}\right\}-\exp\left\{-\frac{(w-m)^{\alpha}}{\theta}\right\}\right)\\ \qquad -\theta^{\frac{1}{\alpha}+1}\left\{\gamma\left(\frac{1}{\alpha}+1,\frac{(w-a)^{\alpha}}{\theta}\right)-\gamma\left(\frac{1}{\alpha}+1,\frac{(w-m)^{\alpha}}{\theta}\right)\right\}\end{array}\right], \quad (b \le w < \infty)$$

(21)

where $\gamma(\alpha, x)$ is the incomplete gamma function given in the book by Erdelyi et al. (1953).

Proof: Substituting the values of $f(x_1)$ and $g(x_2)$ from (20) and (15) in the corollary and simplifying the integrals thus obtained we arrive at the above result (21). Figure 3 illustrates possible shape of the pdf (21) for $a = 0$, $m = 1$, $b = 2$, $\theta = 1$, $\alpha = 1$.

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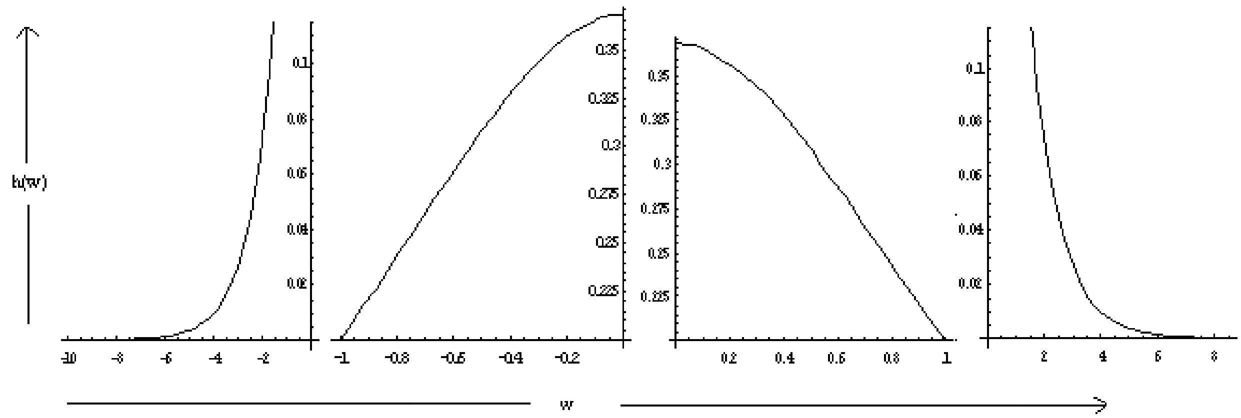
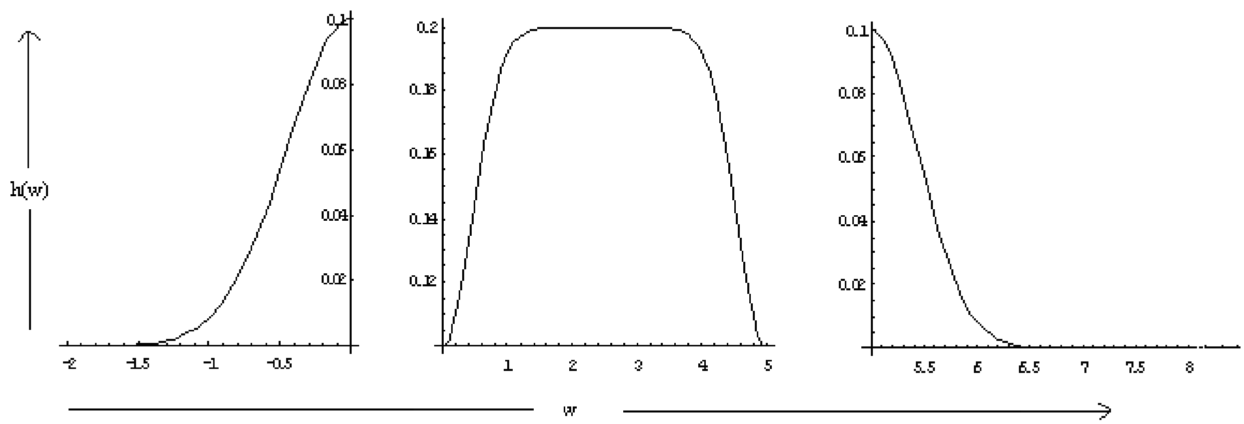
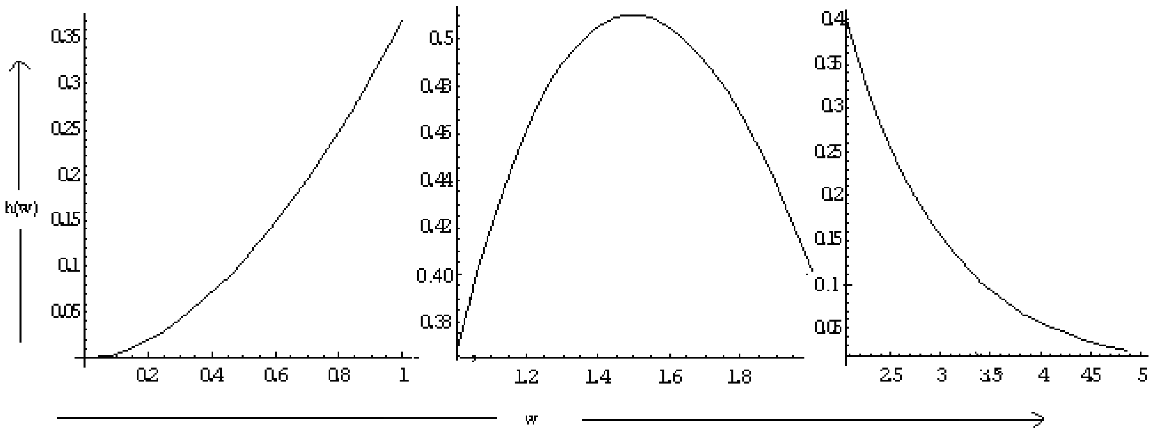


$a = -1$, $m = 0$, $b = 1$
Figure 1



$\theta = 5$, $a = 0$, $b = 5$
Figure 2



$a = 0, m = 1, b = 2, \theta = 1, \alpha = 1$

Figure 3