Applications and Applied Mathematics: An International Journal (AAM)

Volume 2 | Issue 2 Article 4

12-2007

On the Total Duration of Negative Surplus of a Risk Process with Two-step Premium Function

Pavlina Jordanova
Shumen University

Available at
http://pvamu.edu/pages/398.asp
ISSN: 1932-9466
Vol. 2, Issue 2 (December 2007), pp. 107 – 121
(Previously, Vol. 2, No. 2)

Applications and Applied Mathematics (AAM):
An International Journal

On the Total Duration of Negative Surplus of a Risk Process with Two-step Premium Function

Pavlina Jordanova

Faculty of Mathematics and Informatics, Shumen University
115 Universitetska St.
Shumen 9700, Bulgaria
E-mail: pavlina_kj@abv.bg

Received July 4, 2007; accepted September 16, 2007

Abstract

We consider a risk reserve process whose premium rate reduces from c_d to c_u when the reserve comes above some critical value v. In the model of Cramer-Lundberg with initial capital $u \geq 0$, we obtain the probability that ruin does not occur before the first up-crossing of level v. When $u < v$, following H. Gerber and E. Shiu (1997), we derive the probability that starting with initial capital u ruin occurs and the severity of ruin is not bigger than v. Further we express the probability of ruin in the two step premium function model - $\psi(u,v)$, by the last two probabilities. Our assumptions imply that the surplus process will go to infinity almost surely. This entails that the process will stay below zero only temporarily. We derive the distribution of the total duration of negative surplus and obtain its Laplace transform and mean value. As a consequence of these results, under certain conditions in the Model of Cramer-Lundberg we obtain the expected value of the severity of ruin. In the end of the paper we give examples with exponential claim sizes.

Keywords: Surplus process; Probability of ruin; Total duration of negative surplus

AMS 2000 Subject Classification No.: 91B30; 60G99

1. Introduction

In this paper $Y_0 = 0$. We denote by Y_1, Y_2, ... the claim amounts and suppose that they are positive, independent and identically distributed (i.i.d.) random variables (r.vs.) with distribution function (d.f.) F which is absolutely continuous, and with finite mean

$$EY_1 = \frac{1}{\mu} < \infty.$$

We assume that the times between claim arrivals X_1, X_2, ... are independent and exponentially distributed r.vs. with parameter $\lambda > 0$ and the sequences $\{Y_i : i \in \mathbf{N}\}$ and $\{X_i : i \in \mathbf{N}\}$ are independent.

In the classical risk theory usually the time of ruin, the probability of ruin, and the deficit at ruin are examined. A good treatment in this area is Grandel (1991).

In the Cramer-Lundberg model the premiums are received at a constant rate $c > 0$ per unit time. In this case the duration of negative surplus have been investigated in Reis (1993) and in Dickson and Reis (1996). The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin is examined by Gerber and Shiu (1997). The distribution of the time of ruin $\tau_{CL}(u)$ for the cases when the initial capital $u = 0$, in terms of the d.f. of the aggregate claim amount is expressed by the following Seal's formulae (see e.g. Rolski (1999), Theorem 5.6.2.):

$$\mathbf{P}(\tau_{CL}(0) > t) = \frac{1}{ct}\int_0^{ct} \mathbf{P}\Big(\sum_{i=0}^{N_{CL}(t)} Y_i \le y\Big)\, dy; \qquad (1.1)$$

and if $u > 0$ then

$$\mathbf{P}(\tau_{CL}(u) > t) = \mathbf{P}\Big(\sum_{i=0}^{N_{CL}(t)} Y_i \le u + ct\Big) - c\int_0^{u} \mathbf{P}(\tau_{CL}(0) > t - u)\, \mathbf{P}\left(\frac{\sum_{i=0}^{N_{CL}(t)} Y_i - u}{c} \le y\right) dy$$

Here we denote by $N_{CL}(t)$ the number of claim arrivals up to time t.

E. Kolkovska, et al. (2005) prove the existence of local time of renewal risk process with continuous claim distribution. They obtain the Laplace functional of the occupation measure of the risk process.

We consider a particular case of a collective risk model with risk dependent premiums. The risk process is defined by

$$R(t) = u + \int_0^t c(R(s))\, ds - \sum_{i=0}^{N(t)} Y_i, \qquad t \ge 0,$$

where

$$c(y) = \begin{cases} c_d, & y \le v \\ c_u, & y > v \end{cases}$$

and v is a fixed, nonnegative real number.

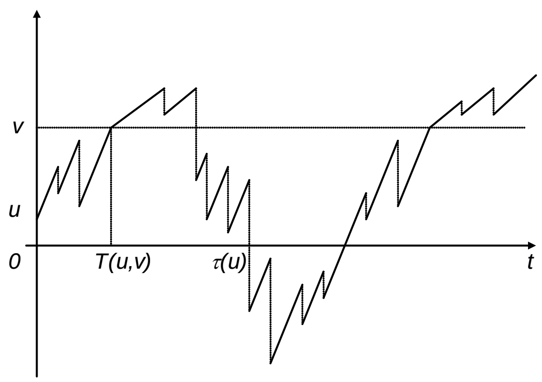

Fig. 1. Case $u < v$.

Such a model is called two-step premium function model (see e.g. Asmussen (1996)). For examples of sample paths see Figure 1 and Figure 2.

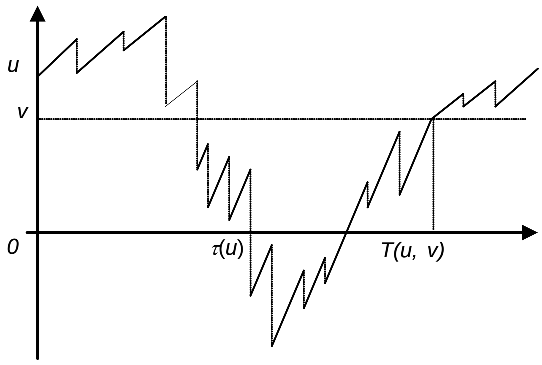

Fig. 2. Case $u \geq v$.

Without loss of generality we assume that $min(c_d, c_u) = c_u$. Our first step will be to require the basic net profit condition

$$min(c_d, c_u) > \frac{EY_1}{EX_1}.$$

This leads to $\psi(u, v) < 1$ for all $0 \leq u < \infty$, where $\psi(u, v)$ is the probability of ruin in the two-step premium function model with initial capital u. Our assumptions imply also that

$$\mathbf{P}(R(t) \to \infty \text{ for } t \to \infty) = 1.$$

The latter means that the duration of a single period of negative surplus and the number of down-crosses of level zero by the surplus process are almost surely finite.

2. Preliminary Results

In this section we consider the Cramer-Lundberg model with initial capital u and premium income rate $c > \frac{EY_1}{EX_1}$.

We denote by

$$F_I(y) = \mu \int_0^y (1 - F(x))\, dx$$

the integrated tail distribution of F, by $\psi_{CL}(u)$ the probability of ruin and by $\delta_{CL}(u)$ the survival probability. The following three formulae can be found in any basic textbook on Risk theory.

$$\delta_{CL}(0) = 1 - \frac{\lambda}{c\mu}, \tag{2.1}$$

$$\delta_{CL}(u) = \delta_{CL}(0) + \int_0^u \delta_{CL}(u - y)\, d_y ((1 - \delta_{CL}(0))\, F_I (y)), \tag{2.2}$$

$$\delta_{CL}(u) = \delta_{CL}(0) + \delta_{CL}(0) \sum_{k=1}^{\infty} (1 - \delta_{CL}(0))^k F_I^{*k}(u). \tag{2.3}$$

Let $q_{CL}(u, v)$ be the probability that ruin does not occur before the first up-crossing of level v.

In this section we assume that $0 \le u < v$.

Because of $\psi_{CL}(u) = 1 - q_{CL}(u, v) + \psi_{CL}(u)\, q_{CL}(u, v)$ (see e.g. Asmussen (1996)) we have

$$q_{CL}(0, v) = \frac{\delta_{CL}(0)}{\delta_{CL}(v)}, \tag{2.4}$$

$$q_{CL}(u, v) = \frac{\delta_{CL}(u)}{\delta_{CL}(v)} = \frac{q_{CL}(0,v)}{q_{CL}(0,u)}. \tag{2.5}$$

In the following lemma we derive an integral equation and explicit formula for $q_{CL}(u, v)$.

Lemma 1:

$$q_{CL}(u, v) = q_{CL}(0, v) + \int_0^u q_{CL}(u - y, v)\, d_y\left((1 - \delta_{CL}(0))\, F_I(y)\right), \tag{2.6}$$

$$q_{CL}(u, v) = \frac{1 + \sum_{k=1}^{\infty}(1-\delta_{CL}(0))^k F_I^{*k}(u)}{1 + \sum_{k=1}^{\infty}(1-\delta_{CL}(0))^k F_I^{*k}(v)}. \tag{2.7}$$

Proof:

We divide (2.2) to $\delta_{CL}(v)$, then we use (2.5) and obtain (2.6). Now we divide (2.3) to $\delta_{CL}(0)$, then we use (2.4) and obtain

$$q_{CL}(0, v) = \frac{1}{1 + \sum_{k=1}^{\infty}(1-\delta_{CL}(0))^k F_I^{*k}(v)}. \tag{2.8}$$

If we gather (2.8) and (2.5) we come to (2.7).

We denote by $G_{CL}(u, v)$ the probability that starting with initial capital u ruin occurs and the severity of ruin is not bigger than v. Then, because F is absolutely continuous $G_{CL}(u, 0) = 0$, for all $u \ge 0$. Let us remind that

$$G_{CL}(0, v) = \psi_{CL}(0)\, F_I(v). \tag{2.9}$$

See e.g. N. Bowers et al. (1987) or the key formula in Gerber and Shiu (1997). They also obtain that starting with initial capital zero, this is the probability that ruin occurs and the surplus immediately before ruin is less than v. It is well known that it coincides with the defective d.f. of the first ascending ladder height of the random walk $S_n = \sum_{k=1}^{n} (Y_k - cX_k)$.

The sum in the denominator of (2.7) is just the renewal function $\sum_{k=1}^{\infty}(G_{CL}(0,v))^{k*}$. As is known it equals the expected value of the number of ascending ladder heights of the corresponding terminating renewal process before the first up-crossing of level v. It is also the mean number of minimums of the risk reserve process with initial capital 0 before its first down-crossing of level $-v$.

Having in mind the paper of Gerber and Shiu (1997) we obtain $G_{CL}(u, v)$.

Lemma 2:

For $u \geq 0$

$$G_{CL}(u, v) = \frac{1}{1-\psi_{CL}(0)}\{\psi_{CL}(u) - \psi_{CL}(u+v) + \int_0^v \psi_{CL}(v+u-z)\, d_z G_{CL}(0, z) - \psi_{CL}(u) G_{CL}(0,v)\}.$$

Proof:

For $u = 0$ by (2.2) and (2.9) the lemma is obviously true. We will consider the case when $u>0$. By $R_{CL}(u, t)$ we denote the risk reserve up to time $t \geq 0$ and by $\tau_{CL}(u)$ the time of ruin. By (4.6) in Gerber and Shiu (1997) for the function $\Gamma(u, x)$ which is defined as the solution of the equation

$$\Gamma(u, x) = \int_0^u \Gamma(u-z)\, d_z G_{CL}(0, z) + I_{\{u<x\}}(x),$$

where

$$I_A(x) = \begin{cases} 1 & , \quad x \in A \\ 0 & \quad elsewhere \end{cases}$$

we have

$$\mathbf{P}_{R_{CL}(u,\tau_{CL}(u)^-),-R_{CL}(u,\tau_{CL}(u))}(x, y;\ \tau_{CL}(u) < \infty) =$$

$$= \mathbf{P}_{R_{CL}(0,\tau_{CL}(0)^-),-R_{CL}(0,\tau_{CL}(0))}(x, y;\ \tau_{CL}(0) < \infty)\Gamma(u, x).$$

By (3.12) in Gerber and Shiu (1997)

$$\mathbf{P}_{R_{CL}(0,\tau_{CL}(0)^-),-R_{CL}(0,\tau_{CL}(0))}(x, y;\ \tau_{CL}(0) < \infty) = \mu.\,\psi_{CL}(0)\, \mathbf{P}_{Y_1}(x+y).$$

This means that

$$\mathbf{P}_{R_{CL}(u,\tau_{CL}(u)^-),-R_{CL}(u,\tau_{CL}(u))}(x, y;\ \tau_{CL}(u) < \infty) = \mu.\,\psi_{CL}(0)\, \mathbf{P}_{Y_1}(x+y)\, \Gamma(u, x). \qquad (2.10)$$

By Dickson's formulae (See Dickson (1992)) and (5.2) in Gerber and Shiu (1997) we have

$$\Gamma(u, x) = \frac{1}{1-\psi_{CL}(0)}\begin{cases} 1-\psi_{CL}(u) & , \quad x > u \geq 0 \\ \psi_{CL}(u-x) - \psi_{CL}(u) & , \quad 0 < x \leq u \end{cases}$$

We substitute this expression in the equation (2.10). So we obtain for $x > u \geq 0$

$$\mathbf{P}_{R_{CL}(u,\tau_{CL}(u)^-),-R_{CL}(u,\tau_{CL}(u))}(x, y;\ \tau_{CL}(u) < \infty) = \frac{\mu.\psi_{CL}(0)(1-\psi_{CL}(u))}{1-\psi_{CL}(0)}\, \mathbf{P}_{Y_1}(x+y) \qquad (2.11)$$

and for $0 < x \leq u$

$$\mathbf{P}_{R_{CL}(u,\tau_{CL}(u)^-),-R_{CL}(u,\tau_{CL}(u))}(x, y;\ \tau_{CL}(u) < \infty) = \frac{\mu.\psi_{CL}(0)(\psi_{CL}(u-x)-\psi_{CL}(u))}{1-\psi_{CL}(0)}\, \mathbf{P}_{Y_1}(x+y). \qquad (2.12)$$

Now we are ready to obtain $G_{CL}(u, v)$. It is true, that

$$G_{CL}(u, v) = \mathbf{P}(\tau_{CL}(u) < \infty, -R_{CL}(u, \tau_{CL}(u)) \leq v) = \int_0^v \mathbf{P}_{-R_{CL}(u,\tau_{CL}(u))}(y;\ \tau_{CL}(u) < \infty)\, dy.$$

Integrating the joint density function with respect to x we obtain the density of the severity of ruin, so

$$G_{CL}(u, v) = \int_0^v \int_0^\infty \mathbf{P}_{R_{CL}(u,\tau_{CL}(u)^-),-R_{CL}(u,\tau_{CL}(u))}(x, y;\ \tau_{CL}(u) < \infty)\, dx\, dy.$$

By changing the order of integration and applying (2.11) and (2.12) we can say that

$$G_{CL}(u, v) = \frac{\mu\psi_{CL}(0)}{1-\psi_{CL}(0)} \Big\{ \int_0^u (\psi_{CL}(u - x) - \psi_{CL}(u))\, \overline{F}(x)\, dx + \int_u^\infty (1 - \psi_{CL}(u))\, \overline{F}(x)\, dx$$

$$- \int_0^u (\psi_{CL}(u - x) - \psi_{CL}(u))\, \overline{F}(x + v)\, dx - \int_u^\infty (1 - \psi_{CL}(u))\, \overline{F}(x + v)\, dx \Big\}$$

$$= \frac{\mu\psi_{CL}(0)}{1-\psi_{CL}(0)} \Big\{ \int_0^u \psi_{CL}(u - x)\, \overline{F}(x)\, dx - \frac{\psi_{CL}(u)F_I(u)}{\mu} + \frac{(1-\psi_{CL}(u))(1-F_I(u))}{\mu}$$

$$- \int_v^{u+v} \psi_{CL}(u + v - x)\, \overline{F}(x)\, dx + \psi_{CL}(u) \int_v^{u+v} \overline{F}(x)\, dx + (1 - \psi_{CL}(u)) \int_{u+v}^\infty \overline{F}(x)\, dx \Big\}$$

$$= \frac{1}{1-\psi_{CL}(0)} \{ \psi_{CL}(0) \int_0^u \psi_{CL}(u - x)\, d_x F_I(x) + \psi_{CL}(0)\, (1 - F_I(u)) - \psi_{CL}(0)\psi_{CL}(u)$$

$$- \int_v^{u+v} \psi_{CL}(u + v - x)\, d_x\, (\psi_{CL}(0)\, F_I(x)) + \psi_{CL}(0)\psi_{CL}(u)\, (1 - F_I(u)) - \psi_{CL}(0)\, (1 - F_I(u + v))\}.$$

Because of (2.9) and (2.2)

$$G_{CL}(u, v) = \frac{1}{1-\psi_{CL}(0)} \{ \psi_{CL}(u) - \psi_{CL}(u + v) + \int_0^v \psi_{CL}(v + u - z)\, d_z\, G_{CL}(0, z)$$

$$- \psi_{CL}(u) G_{CL}(0,v)\}.$$

So we completed the proof.

3. The Total Duration of Negative Surplus

In the classical Cramer-Lundberg model with initial capital $u = 0$, Reis (1993) notes that, given that ruin occurs, the duration of the first single period of a negative surplus (we denote it by $\eta_{1,CL}(0)$) coincides in distribution with the time of ruin. It is also identically distributed with the other periods of negative surpluses, conditionally that they are positive. This distribution could be found by Seal's formula (1.1). For $t \geq 0$

$$\mathbf{P}(\tau_{CL}(0) \leq t \mid \tau_{CL}(0) < \infty) = \frac{1}{\psi_{CL}(0)} \{1 - \frac{1}{ct} \int_0^{ct} P(\sum_{i=0}^{N_{CL}(t)} Y_i \leq y) dy\}.$$

In the model of Cramer-Lundberg we define $T_{CL}(u, v)$ to be the time of the first up-crossing of level v and as before, $G_{CL}(u, y)$ to be the probability that starting with initial capital u ruin occurs and the severity of ruin is not bigger than y. By Dickson and Reis (1996), for arbitrary initial capital $u > 0$

$$\mathbf{P}(\eta_{1,CL}(u) \leq t \mid \eta_{1,CL}(u) > 0) = \frac{1}{\psi_{CL}(u)} \int_0^{ct} P(T_{CL}(0, y) \leq t) d_y G_{CL}(u, y), \qquad t \geq 0.$$

In the two-step premium function model with initial capital $u \ge 0$ the single sojourn time of the risk reserve process in $(-\infty, 0]$ could be considered like the single sojourn time in $(-\infty, 0]$ of the risk reserve process in the model of Cramer-Lundberg with premium income rate c_d. For all k = 1, 2, ..., given that the duration of k-th negative surplus is positive it has the same distribution whether we consider the model of Cramer-Lundberg with premium income rate c_d or the two step premium function model. Further instead of lower index $_{CL}$ we will write $_d$ when the discussed quantity is equal to the certain quantity in the model of Cramer-Lundberg with premium income rate c_d. Analogously we use lower index $_u$.

Let $\eta_k(u)$, k =1, 2, ... be the duration of k-th negative surplus in the two-step premium functions model with initial capital u. Given that $\eta_1(u)$, $\eta_2(u)$,... are positive they are independent and $\eta_2(u)$, $\eta_3(u)$, ... are identically distributed.

$$\mathbf{P}(\eta_1(0) \le x \mid \eta_1(0) > 0) = \mathbf{P}_{\mathrm{d}}(\eta_1(0) \le x \mid \eta_1(0) > 0) \tag{3.1}$$

for $x \in \mathbf{R}$.

When $u > 0$

$$\mathbf{P}(\eta_1(u) \le x \mid \eta_1(u) > 0) = \mathbf{P}_{\mathrm{d}}(\eta_1(u) \le x \mid \eta_1(u) > 0) \tag{3.2}$$

for $x \in \mathbf{R}$.

When k = 2, 3, ... and $u \ge 0$

$$\mathbf{P}(\eta_k(u) \le x \mid \eta_k(u) > 0) = \mathbf{P}(\eta_1(0) \le x \mid \eta_1(0) > 0), \qquad x \in \mathbf{R}. \tag{3.3}$$

We assume that $\eta_0 = 0$. The total duration of negative surplus $\eta(u, v)$ in the two-step premium functions model can be presented as random sum

$$\eta(u, v) = \sum_{i=0}^{N(u,v,0)} \eta_i(u),$$

where $N(u, v, 0)$ is the number of occasions on which the surplus process falls below zero.

Let us note that $N(u, v, 0)$ and $\eta_1(u)$, $\eta_2(u)$,... are not independent. To come to the distribution of $\eta(u, v)$ we have to determine the distribution of $N(u, v, 0)$.

It is not difficult to obtain that

$$\mathbf{P}(N(u, v, 0) = 0) = 1 - \psi(0, v), \tag{3.4}$$

and for k = 1,2,...

$$\mathbf{P}(N(u, v, 0) = k) = \psi(u, v)\, \psi^{k-1}(0, v)(1 - \psi(0, v)). \tag{3.5}$$

In the following theorem we express $\psi(u, v)$ in different cases for u and v.

Theorem 1:

For the two-step premium function model with net-profit condition (2.1)

a) If $u = v$, then

$$\psi(v,v) = \frac{\psi_u(0)\psi_d(v)\delta_d(0)}{\delta_d(v)\psi_d(0) - \psi_u(0)\delta_d(v) + \psi_u(0)\delta_d(0)}; \tag{3.6}$$

b) If $u < v$, then

$$\psi(u,v) = 1 - \frac{\psi_d(0)\delta_d(u)\delta_u(0)}{\delta_d(v)\psi_d(0) - \psi_u(0)\delta_d(v) + \psi_u(0)\delta_d(0)}; \qquad (3.7)$$

c) If $u > v$, then

$$\psi(u,v) = \psi_u(u - v) - \frac{\delta_u(0)\psi_d(0)\int_0^v \delta_d(v-y)d_y G_u(u-v,y)}{\delta_d(v)\psi_d(0) - \psi_u(0)\delta_d(v) + \psi_u(0)\delta_d(0)}. \qquad (3.8)$$

Here $\psi_u(u)$ is the probability of ruin in the Cramer-Lundberg model with initial capital u and premium income rate per unit time c_u, $G_u(u, v)$ is equal to $G_{CL}(u, v)$ and $\delta_u(u) = 1 - \psi_u(u)$. $\delta_d(u)$ is the survival probability in the model of Cramer-Lundberg with initial capital u and premium income rate c_d.

Proof:

We denote by $R_u(x, t)$ the risk reserve up to time $t \geq 0$ in the model of Cramer-Lundberg with initial capital $x \geq 0$ and premium income rate $c_u > 0$. We define $\tau_u(x)$ to be the time of ruin, $T_u(x, v)$ - the time of the first up-crossing of level v and $q_u(x, v)$ to be the probability that ruin does not occur before the first up-crossing of level v.

Analogously we denote by $R_d(x, t)$ the risk reserve up to time $t \geq 0$ in the model of Cramer-Lundberg with initial capital $x \geq 0$ and premium income rate $c_d > 0$. $\tau_d(x)$ is the time of ruin in this model, and $\psi_d(x)$ is the probability of ruin. Let $T_d(x, v)$ be the time of the first up-crossing of level v.

In the two step premium function model with initial capital u and critical level x we define $\theta(u, x, t)$ to be the number of down-crossings of level v up to time t.

We consider the following three groups of events:
$A(u, v)$ = ”starting with initial capital u there is no ruin before the first up-crossing of level v”,
$B(u, v)$ = ” $\tau(u) < \infty$, $\theta(u, v, \tau(u)) = 1$ and the time of the down-crossing of the level v coincides with the time of ruin” and
$C(u, v)$ = ” $\tau(u) < \infty$, $\theta(u, v, \tau(u)) = 1$ and the time of the down-crossing of the level v does not coincide with the time of ruin”.

a) At this point we suppose that the risk reserve process starts with initial capital v. By the Theorem of Total Probability we have

$$\psi(v, v) = \sum_{i=1}^{\infty} \mathbf{P}\,(\tau(v) < \infty,\ \theta(v, v, \tau(v)) = k) \qquad (3.9)$$

To find $\mathbf{P}\,(B(v, v))$ we consider only the risk-reserve process $R_u(0, t)$. It is not difficult to realize, that

$$\mathbf{P}(B(v, v)) = \mathbf{P}(\tau_u(0) < \infty, -R_u(0,\tau_u(0)) > v) = \psi_u(0) - \mathbf{P}(\tau_u(0) < \infty, -R_u(0,\tau_u(0)) \leq v).$$

By (2.9)

$$\mathbf{P}(B(v, v)) = \psi_u(0) - \psi_u(0)F_I(v) = \psi_u(0)(1 - F_I(v)) = \psi_u(0)\overline{F}_I(v). \qquad (3.10)$$

By the Theorem of Total Probability and (2.9) we obtain

$$\mathbf{P}(C(v, v)) = \int_0^v \mathbf{P}(T_d(v-y, v) > \tau_d(v-y))\, d_y\, \mathbf{P}(-R_u(0, \tau_u(0)) \le y, \tau_u(0) < \infty)$$

$$= \psi_u(0) \int_0^v (1 - q_d(v-y, v))\, d_y\, F_I(y).$$

In view of (2.5), a formula equivalent to the above one is

$$\mathbf{P}(C(v, v)) = \psi_u(0).F_I(v) - \psi_u(0) \int_0^v \frac{\delta_d(v-y)}{\delta_d(v)}\, d_y\, F_I(y).$$

Now we use (2.2) and obtain

$$\mathbf{P}(C(v, v)) = \psi_u(0).F_I(v) - \psi_u(0) \frac{\delta_d(v) - \delta_d(0)}{\delta_d(v)\psi_d(0)}.$$

By distinguishing whether or not ruin occurs at the first time when the surplus falls below the initial capital v, the law of total probability yields

$$\mathbf{P}(\tau(v) < \infty, \theta(v, v, \tau(v)) = 1) = \mathbf{P}(B(v, v)) + \mathbf{P}(C(v, v))$$

$$= \psi_u(0) - \psi_u(0) \frac{\delta_d(v) - \delta_d(0)}{\delta_d(v)\psi_d(0)} = \frac{\psi_u(0)\delta_d(0)\psi_d(v)}{\delta_d(v)\psi_d(0)}.$$

Let us find $\mathbf{P}(\tau(v) < \infty, \theta(v, v, \tau(v)) = 2)$. Then

$$\mathbf{P}(A(v, v)) = \int_0^v \mathbf{P}(T_d(v-y, v) \le \tau_d(v-y))\, d_y\, \mathbf{P}(-R_u(0, \tau_u(0)) \le y, \tau_u(0) < \infty).$$

Analogously to $\mathbf{P}(C(v, v))$ we obtain

$$\mathbf{P}(A(v, v)) = \psi_u(0) \int_0^v q_d(v-y, v))\, d_y\, F_I(y) = \psi_u(0) \frac{\delta_d(v) - \delta_d(0)}{\delta_d(v)\psi_d(0)}.$$

By distinguishing whether or not ruin occurs at the second time when the surplus falls below the initial capital v and applying the law of total probability, we obtain

$$\mathbf{P}(\tau(v) < \infty, \theta(v, v, \tau(v)) = 2) = \mathbf{P}(A(v, v))(\mathbf{P}(B(v, v)) + \mathbf{P}(C(v, v))).$$

Analogously

$$\mathbf{P}(\tau(v) < \infty, \theta(v, v, \tau(v)) = k) = \mathbf{P}^{k-1}(A(v, v))(\mathbf{P}(B(v, v)) + \mathbf{P}(C(v, v))).$$

Finally applying (3.9) we can calculate

$$\psi(v, v) = \sum_{k=1}^{\infty} \mathbf{P}^{k-1}(A(v, v))(\mathbf{P}(B(v, v)) + \mathbf{P}(C(v, v)))$$

$$= \frac{\mathrm{P(B}(v, v)) + \mathrm{P(C}(v, v))}{1 - \mathrm{P(A}(v, v))} = \frac{\psi_u(0) + \mathrm{P(A}(v, v))}{1 - \mathrm{P(A}(v, v))}$$

$$= 1 - \frac{\delta_u(0)\delta_d(v)\psi_d(0)}{\delta_d(v)\psi_d(0) - \psi_u(0)\delta_d(v) + \psi_u(0)\delta_d(0)}$$

$$= \frac{\psi_u(0)\psi_d(v)\delta_d(0)}{\delta_d(v)\psi_d(0) - \psi_u(0)\delta_d(v) + \psi_u(0)\delta_d(0)}.$$

b) Analogously to Asmussen (1996) we obtain $\psi(u, v) = 1 - q_d(u, v) + \psi(v, v)\, q_d(u, v)$. Now, we replace $\psi(v, v)$ with the expression in **a)** and $q_d(u, v)$ by (2.5) and come to (3.7).

c) As above, by distinguishing whether or not ruin occurs at the first time, when the surplus falls below the critical level v, or there is no ruin before the first up-crossing of level v and applying the law of total probability, we obtain

$$\psi(u, v) = \mathbf{P}(B(u, v)) + \mathbf{P}(C(u, v)) + \mathbf{P}(A(u, v))\, \psi(v, v). \tag{3.11}$$

We found $\psi(v, v)$ in **a)**. To find $\mathbf{P}(B(u, v))$ we consider an auxiliary risk-reserve process R_u with initial capital $u - v$ and premium income rate c_u.

$$\begin{aligned}\mathbf{P}(B(u, v)) &= \mathbf{P}(\tau_u(u - v) < \infty, -R_u(u - v, \tau_u(u - v)) > v)\\ &= \psi_u(u - v) - \mathbf{P}(\tau_u(u - v) < \infty, -R_u(u - v, \tau_u(u - v)) \le v)\\ &= \psi_u(u - v) - G_u(u - v, v).\end{aligned}$$

By Total probability, (2.5) and (2.2)

$$\begin{aligned}\mathbf{P}(C(u, v)) &= \int_0^v (1 - q_d(v - y, v))\, d_y G_u(u - v, y)\\ &= G_u(u - v, v) - \int_0^v \frac{\delta_d(v - y)}{\delta_d(v)}\, d_y G_u(u - v, y),\end{aligned}$$

and

$$\mathbf{P}(A(u, v)) = \int_0^v q_d(v - y, v)\, d_y G_u(u - v, y) = \int_0^v \frac{\delta_d(v - y)}{\delta_d(v)}\, d_y G_u(u - v, y).$$

Substituting these expressions in (3.11) and using (2.5), we obtain

$$\begin{aligned}\psi(u, v) &= \psi_u(u - v) - \mathbf{P}(A(u, v))\, \delta(v, v)\\ &= \psi_u(u - v) - \frac{\delta_u(0)\psi_d(0)\int_0^v \delta_d(v - y)\, d_y G_u(u - v, y)}{\delta_d(v)\psi_d(0) - \psi_u(0)\delta_d(v) + \psi_u(0)\delta_d(0)}.\end{aligned}$$

Note: 1. As a consequence we obtained the obvious result that for $u \ge 0$, it is true that

$$\psi(u, 0) = \psi_u(u).$$

2. It is interesting to note, that if $u \le v$ then

$$\frac{\psi(u,v) - \psi(v,v)}{\psi_d(u) - \psi_d(v)} = \frac{\psi(0,v) - \psi(v,v)}{\psi_d(0) - \psi_d(v)} = \frac{1}{1 - \psi_d(v)(1 - \frac{\rho_d}{\rho_u})}.$$

3. If we compare the two - step premium function model with initial capital u and critical level $v \le u$ and the Model of Cramer -Lundberg with initial capital $u - v$ and premium income rate c_u it is interesting to mention that the following relation holds

$$\psi_u (u - v) = \mathbf{P}(A(u, v)) + \mathbf{P}(B(u, v)) + \mathbf{P}(C(u, v)).$$

Now we are ready to obtain our main result.

Theorem 2:

For the two-step premium function model with net-profit condition (2.1), initial capital $u \ge 0$ and critical level $v \ge 0$

i) for $x \le 0$, $\mathbf{P}(\eta (u, v) \le x) = 0$ and for $x \ge 0$,

$$\mathbf{P}(\eta (u, v) < x) = (1 - \psi (0, v)) + \psi (u, v)(1 - \psi (0, v)) \sum_{i=1}^{\infty} K_1 * K_2 *^{(k - 1)} (x)(\psi (0, v))^{k - 1}, \quad (3.12)$$

where $K_1 (x) = \mathbf{P}(\eta_1(u) \le x \mid \eta_1(u) > 0)$ and $K_2 (x) = \mathbf{P}(\eta_1(0) \le x \mid \eta_1(0) > 0)$ are determined by (3.1), (3.2) and (3.3).

ii) for x > 0

$$Ee^{-x\eta(u, v)} = (1 - \psi (0, v)) \left(1 + \frac{\psi(u,v)\dfrac{1}{\psi_d(u)}\displaystyle\int_0^\infty e^{-y\, f(x)} d_y G_d(u,y)}{1-\psi(0,v)\dfrac{\mu(Ee^{-Y_1 f(x)}-1)}{-f(x)}}\right),$$

where the function $f(s)$ satisfies the equation $s = c_d f (s) + \lambda (E\, e^{-Y1\ f(s)} - 1)$, $s < 0$.

iii) If $DY_1 < 1$, then

$$E\eta (u, v) = \frac{\psi(u,v)}{c_d \delta_d(0)}\left(\frac{1}{\psi_d(u)}\int_0^\infty x\ d_x G_d(u,x) + \frac{\mu\ E(Y_1^{\ 2})\psi(0,v)}{2\delta(0,v)}\right).$$

Proof:

i) By total probability (3.4), (3.5) and

$$\mathbf{P}(\eta(u, v) < x) = \mathbf{P}\left(\sum_{i=1}^{N(u,v,0)} \eta_i\ (u) < x\right),$$

we have (3.12).

ii) By (3.6) in Dickson and Reis (1996) for all $u > 0$ and for all $x > 0$

$$E_d(e^{-x\eta_1(u)} \mid \eta_1(u) > 0) = \frac{1}{\psi_d(u)}\int_0^\infty e^{-y\ f(x)} d_y G_d(u,y),$$

where the function $f(s)$ satisfies the equation $s = c_d f (s) + \lambda (E\, e^{-Y1\ f(s)} - 1)$, for $s < 0$.

By Reis (1993) we have for $k = 2,3,...$

$$E_d (e^{-x\eta_k(u)} \mid \eta_k (u) > 0) = E_d(e^{-x\eta_1(0)} \mid \eta_1 (0) > 0) = \frac{\mu(Ee^{-Y_1 f(x)}-1)}{-f(x)},$$

where $f(s)$ is the same function as above.

These formulae and the law of the total expectation yield

$$Ee^{-x\eta(u,v)} = E\,e^{-x\sum_{i=1}^{N(u,v,0)}\eta_i(u)}$$

$$= 1 - \psi(0,v) + \psi(u,v)(1-\psi(0,v)).E(e^{-x\eta_1(u)}\,|\,\eta_1(u)>0)$$

$$.\sum_{k=1}^{\infty} \psi^{k-1}(0,v)(E(e^{-x\eta_1(0)}\,|\,\eta_1(0)>0))^{k-1}$$

$$= (1-\psi(0,v))\left(1+\frac{\psi(u,v).E(e^{-x\eta_1(u)}\,|\,\eta_1(u)>0)}{1-\psi(0,v).E(e^{-x\eta_1(0)}\,|\,\eta_1(0)>0)}\right)$$

$$= (1-\psi(0,v))\left(1+\frac{\psi(u,v)\dfrac{1}{\psi_d(u)}\displaystyle\int_0^{\infty} e^{-y\,f(x)}d_yG_d(u,y)}{1-\psi(0,v)\dfrac{\mu(Ee^{-Y_1 f(x)}-1)}{-f(x)}}\right).$$

iii) By Reis (1993), we have for all $u > 0$

$$E_d\,(\eta_1(u)\,|\,\eta_1(u)>0) = \frac{1}{c_d\delta_d(0)\psi_d(u)}\int_0^{\infty} x\ d_x\,G_d(u,x).$$

If the variance of Y_1 is finite, by Reis (1993) we have for $k = 2,3,...$

$$E_d\,(\eta_k(u)\,|\,\eta_k(u)>0) = \frac{\mu\ E(Y_1^2)}{2c_d\delta_d(0)}.$$

By these formulae and the law of the total expectation we obtain

$$E\eta(u,v) = E\sum_{i=1}^{N(u,v,0)}\eta_i\,(u) = \sum_{i=1}^{\infty}(\sum_{k=1}^{i} E\,(\eta_k(u)\,|\,\eta_k(u)>0))\psi(u,v)(1-\psi(0,v))\ \psi^{i-1}(0,v)$$

$$=\psi(u,v)(1-\psi(0,v))\sum_{i=1}^{\infty}(\frac{1}{c_d\delta_d(0)\psi_d(u)}\int_0^{\infty} x\ d_xG_d(u,x)+\frac{(i-1)\mu\ E(Y_1^2)}{2c_d\delta_d(0)})\ \psi^{i-1}(0,v)$$

$$=\psi(u,v)(1-\psi(0,v))\left(\frac{1}{(1-\psi(0,v))c_d\delta_d(0)\psi_d(u)}\int_0^{\infty} x\,d_xG_d(u,x)\right.$$

$$\left.+\frac{\mu.E(Y_1^2)}{2c_d\delta_d(0)(1-\psi(0,v))^2}-\frac{\mu.E(Y_1^2)}{2c_d\delta_d(0)(1-\psi(0,v))}\right)$$

$$=\frac{\psi(u,v)}{c_d\delta_d(0)}\left(\frac{1}{\psi_d(u)}\int_0^{\infty} x\ d_xG_d(u,x)+\frac{\mu\ E(Y_1^2)}{2(1-\psi(0,v))}-\frac{\mu\ E(Y_1^2)}{2}\right)$$

$$=\frac{\psi(u,v)}{c_d\delta_d(0)}\left(\frac{1}{\psi_d(u)}\int_0^{\infty} x\ d_xG_d(u,x)+\frac{\mu\ E(Y_1^2)\psi(0,v)}{2\delta(0,v)}\right).$$

4. A Numerical Example

Let

$$\mathbf{P}(Y_1 < x) = \begin{cases} 1 - e^{-\mu x}, & x > 0 \\ 0, & x \le 0 \end{cases}.$$

As it is known, in this case the integrated tail distribution of F is also exponential with the same parameter μ, $\rho = \frac{c\mu}{\lambda} - 1$ and

$$\delta_{CL}(u) = 1 - \frac{e^{-v\mu\frac{\rho}{1+\rho}}}{1+\rho}.$$

According to the above lemmas

$$q_{CL}(0, v) = \frac{\rho}{1+\rho-e^{-v\mu\frac{\rho}{1+\rho}}},$$

and

$$q_{CL}(u, v) = \frac{1+\rho-e^{-u\mu\frac{\rho}{1+\rho}}}{1+\rho-e^{-v\mu\frac{\rho}{1+\rho}}}.$$

Because of $\psi_{CL}(0) = \frac{1}{1+\rho}$ and by (2.9)

$$G_{CL}(0, v) = \frac{1-e^{-v\mu}}{1+\rho}.$$

When we substitute these expressions in Lemma 2 we obtain

$$G_{CL}(u, v) = \frac{e^{-u\mu\frac{\rho}{1+\rho}}}{1+\rho}(1-e^{-\mu v})$$

So we obtain the well known result that in this case

$$G_{CL}(u, v) = \psi_{CL}(u).\ F_I(v). \tag{4.1}$$

Let us remind, that (4.1) is not correct for any claim size distribution function F. In general case, as is shown in Bowers et al. (1987) or in Gerber and Shiu (1997), (4.1) is correct only for $u = 0$.

When we apply Theorem 1 for the exponential claim sizes, we obtain

a) If $u \le v$ then

$$\psi(u,\ v) = \frac{\psi_d(u) - \psi_d(v)(1-\frac{\rho_d}{\rho_u})}{1-\psi_d(v)(1-\frac{\rho_d}{\rho_u})},$$

where $\rho_d = \frac{c_d\mu}{\lambda}$ -1, $\rho_u = \frac{c_u\mu}{\lambda}$ -1 and

$$\psi_d(v) = \frac{e^{-v\mu\frac{\rho_d}{1+\rho_d}}}{1+\rho_d};$$

b) If $u > v$

$$\psi(u,\ v) = \frac{\psi_u(u-v)\delta_d(0)\psi_d(v)}{\delta_d(v)(\psi_u(0)-\psi_d(0))+\psi_d(0)\delta_u(0)}$$

$$= e^{-\mu(u-v)\frac{\rho_u}{1+\rho_u}} \frac{\psi_d(v)\dfrac{\rho_d}{\rho_u}}{1-\psi_d(v)(1-\dfrac{\rho_d}{\rho_u})}$$

$$= e^{-\mu(u-v)\frac{\rho_u}{1+\rho_u}} \psi(v,\ v).$$

It is interesting to note that when the claim sizes are exponentially distributed and $u > v$

$$\frac{\psi(u,v)}{\psi(v,v)} = \frac{\psi_u(u)}{\psi_u(v)}.$$

Let $\lambda = 1$ and $\mu = 4$. By net profit condition (2.1) $min(c_d,\ c_u)$ should be greater than 0.25. Table 1 and Table 2 show values of $\psi(u, v)$ and $E\eta(u, v)$ for different cases of c_u, c_d, u and v.

Table 1. Values of $\psi(u, v)$ and $E\eta(u, v)$ for $c_u = 0.26$ and $c_d = 0.3$; 0.35 or 0.4

c_u	c_d	u	v	$\psi(u, v)$	$E\eta(u, v)$
0.26	0.3	1	1	0.7898	947.9360
			10	0.4303	164.3766
			100	0.4278	162.5821
		10	1	0.1976	237.3822
			10	0.0053	2.0173
			100	0.0011	0.4030
	0.35	1	1	0.7468	281.5653
			10	0.2278	20.5082
			100	0.2278	20.5011
		10	1	0.1870	70.5096
			10	$10^{-5}7.7710$	0.0070
			100	$10^{-6}7.7715$	$10^{-4}6.9944$
	0.4	1	1	0.7085	125.1359
			10	0.1359	5.2685
			100	0.1359	5.2684
		10	1	0.1774	31.3365
			10	$10^{-6}2.8678$	$10^{-4}1.0834$
			100	$10^{-7}1.9119$	$10^{-6}7.2227$

Table 2. Values of $\psi(u, v)$ and $E\eta(u, v)$ for $c_u = 0.3$ and $c_d = 0.3$; 0.4 or 0.5

c_u	c_d	u	v	$\psi(u, v)$	$E\eta(u, v)$
0.3	0.3	1	1	0.4278	162.5821
			10	0.4278	162.5821
			100	0.4278	162.5821
		10	1	0.0011	0.4030
			10	0.0011	0.4030
			100	0.0011	0.4030
	0.4	1	1	0.3271	18.8464
			10	0.1395	5.2684
			100	0.1395	5.2684
		10	1	$10^{-4}8.1087$	0.0467
			10	$10^{-7}5.7357$	$10^{-5}2.1668$
			100	$10^{-7}1.9119$	$10^{-6}7.2227$
	0.5	1	1	0.2663	5.5015
			10	0.0677	0.8120
			100	0.0677	0.8120
		10	1	$10^{-4}6.6001$	0.0136
			10	$10^{-9}5.1529$	$10^{-8}6.1835$
			100	$10^{-9}1.0306$	$10^{-8}1.2367$

Let us remind, that when $c_u = c_d$ our model coincides with the Model of Cramer-Lundberg. In this case, v has no effect neither on the probability of ruin nor on the expected value of the total duration of negative surplus.

When we compare rows 4, 7, 16 and 17 from Table 1 crrespondingly with rows 4, 7, 10 and 13 from Table 2 we can see that in the two-step premium function model, when the critical level v is large enough, the calculated values do not depend on c_u. The reason for this is that in this case the probability of the the event "the risk reserve process will rich this critical level before the last down-crossing of zero level" becames as smaller as v decreases.

Acknowlegement:

The paper is partially supported by NSFI-Bulgaria, Grant No. VU-MI-105/2005. I wish to thank to anonymous referees for their remarks.

REFERENCES

Asmussen, S. (1996). *Ruin Probabilities,* World Scientific, Singapore.

Bowers, N.I., H.U. Gerber, C.J. Hickman, D.A. Jones and C.J. Nesbitt, (1987). *Actuarial Mathematics,* Society of Actuaries, Itaca, IL.

Dickson, D.C.M. (1992). On the Distribution of the Surplus Prior to Ruin, *Insurance: Mathematics and Economics,* **11**, pp.191–207.

Dickson D.C.M. and A. D. Egidio dos Reis, (1996). On the Distribution of the Duration of Negative Surplus, *Scand. Actuarial Journal,* **2**, pp.148–164.

Egidio dos Reis, A. D. (1993). How Long is the Surplus Below Zero?, *Insurance: Mathematics and Economics,* **12**, pp.23–38.

Gerber, H.U. and E. S. W. Shiu, (1997). The Joint Distribution of the Time of Ruin, the Surplus Immediately Before Ruin, and the Deficit at Ruin, *Insurance: Mathematics and Economics*, **21**, pp. 129–137.

Grandel, J. (1991). *Aspects of Risk Theory*, Springer, Berlin.

time of classical risk process, *Insurance: Mathematics and Economics*, **37**, pp 573 - 584. Kolkovska, E., J.O. Lopes-Mimbela and J.V. Morales, (2005). Occupation measure and local

Rolski, T., H. Schmidli, V. Schmidt and J. Teugels, (1999). Stochastic Processes for Insurance and Finance, John Wiley, Chichester.