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STABILITY ANALYSIS FOR AN SEIR AGE-STRUCTURED EPIDEMIC MODEL UNDER VACCINATION

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Abstract

An SEIR age-structured epidemic model is investigated when susceptible and immune individuals are vaccinated indiscriminately and the force of infection of proportionate mixing type. We determine the steady states and obtain an explicitly computable threshold condition, and then study the stability of the steady states.

Keywords: Age-structure; Epidemic; Steady state; Stability; Proportionate mixing.
MSC 2000: 45K05; 45M10; 92D30; 92D25.

1 Introduction

Several recent papers have dealt with age-dependent vaccination models, where age is the chronological age i.e. the elpase of time since birth, for example, Hethcote (1983), (1989),

(1997), (2000), Dietz, et al. (1985), Dietz (1981), Katzmann, et al. (1984), Schenzle (1984), Anderson, et al. (1999), El-Doma (2000), (2001), Müller (1994), (1998), Hadeler, et al. (1996), Knox (1980), McLean (1986), Coutinho, et al. (1993), Lopez, et al. (2000), Greenhalgh (1990), (1988), Castillo-Chavez, et al. (1998), Li, et al. (2004), and Thieme (2001). We note that some vaccines may wane over time which give rise to SIRS type epidemic models considered in Li, et al. (2004) and El-Doma (2006).

In Thieme (2001), an SEIR age-structured epidemic model is investigated when susceptible and immune individuals are vaccinated indiscriminately, and vaccination provides permanent immunity to the disease. Under suitable conditions, the uniform weak persistence of the disease is proved and conditions for the extinction of the disease are derived.

In Li, et al. (2001), an SEIR age-structured epidemic model is investigated without assuming vaccination for individuals in the population; also, $\gamma(a), \eta(a)$, which are, respectively, the cure rate and the rate at which exposed individuals become infective, are assumed to be constants independent of age. However, the force of infection takes a general form which includes proportionate mixing. Local stability results are given in terms of the spectral radius of an operator and so, in contrast to our results in this paper, their thresholds are not explicitly computable. Furthermore, they do not obtain global stability results.

To the best of our knowledge, Thieme (2001) and Li, et al. (2001), are the only two papers in the literature that have dealt with SEIR type epidemic models when age is also considered.

In this paper, we consider the same model as in Thieme (2001), and determine the steady states by proving a threshold theorem and obtain an explicitly computable threshold parameter R_ν, known as the reproduction number in the presence of vaccination strategy ν as in Hadeler, et al. (1996) or the net replacement ratio as in Thieme (2001). If $R_\nu \leq 1$, then the only steady state is the disease-free equilibrium, and we show that this steady state is globally stable if $R_\nu < 1$. If $R_\nu > 1$, then a disease-free equilibrium and an endemic equilibrium are possible steady states, we prove that the disease-free equilibrium is unstable if $R_\nu > 1$ and the endemic equilibrium, under suitable conditions, is locally asymptotically stable, whenever it exists. Also, in some special cases, we prove that the endemic equilibrium is globally stable.

The organization of this paper is as follows: in section 2 we describe the model and obtain the model equations; in section 3 we reduce the model equations to several subsystems; in section 4 we determine the steady states; in section 5 we study the stability of the steady states; and in section 6 we conclude our results.

2 The Model

In this section, we consider an age-structured population of variable size exposed to a communicable disease which causes so few fatalities that they can be neglected. We

assume the following:

1. $s(a,t), e(a,t), i(a,t)$ and $r(a,t)$, respectively, denote the age density for susceptible, exposed, infective and immune individuals of age a at time t. Then
   $\int_{a_1}^{a_2} s(a,t)da$ = total number of susceptible individuals at time t of ages between a_1 and a_2,

   $\int_{a_1}^{a_2} e(a,t)da$ = total number of exposed individuals at time t of ages between a_1 and a_2, and similarly for $i(a,t)$ and $r(a,t)$. We assume that the total population consists entirely of susceptible, exposed, infective and immune individuals.

2. Let $k(a,a')$ denotes the probability that a susceptible individual with age a is infected by an infective individual with age a'. We further assume that $k(a,a') = k_1(a)k_2(a')$, which is known as "proportionate mixing assumption" (see Dietz, el al. (1985)). Therefore the transmission of the disease occurs at the following rate:

   $$k_1(a)s(a,t)\int_0^\infty k_2(a')i(a',t)da' \ ,$$

   where $k_1(a)$, and $k_2(a)$ are bounded, non-negative, continuous functions of a. The term

   $$k_1(a)\int_0^\infty k_2(a')i(a',t)da' \ ,$$

   is called "force of infection" and we let $\lambda(t) = \int_0^\infty k_2(a')i(a',t)da'$.

3. The death rate $\mu(a)$ is the same for susceptible, exposed, infective and immune individuals and $\mu(a)$ is a non-negative, continuous function and $\exists\ a_0 \in [0,\infty)$ such that $\mu(a) > \bar{\mu} > 0\ \forall\ \ a > a_0$ and $\mu(a_2) > \mu(a_1)\ \forall\ \ a_2 > a_1 > a_0$.

4. All offspring are susceptible, i.e. $s(0,t) = B =$ constant, $e(0,t) = 0$, $i(0,t) = 0$ and $r(0,t) = 0$.

5. The cure rate $\gamma(a)$ is a bounded, non-negative, continuous function of a.

6. The vaccination rate $\nu(a)$ is a bounded, non-negative, continuous function of a.

7. The exposed individuals become infective at a rate $\eta(a)$ which is a bounded, non-negative, continuous function of a.

8. The initial age distributions: $s(a,0) = s_0(a), e(a,0) = e_0(a), i(a,0) = i_0(a)$ and $r(a,0) = r_0(a)$ are continuous, non-negative and integrable functions of $a \in [0,\infty)$.

These assumptions lead to the following system of nonlinear integro-partial differential equations, which describes the dynamics of the transmission of the disease:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a} + \dfrac{\partial s(a,t)}{\partial t} + [\mu(a)+\nu(a)]s(a,t) = -k_1(a)s(a,t)\lambda(t), \quad a>0, t>0, \\ \dfrac{\partial e(a,t)}{\partial a} + \dfrac{\partial e(a,t)}{\partial t} + [\mu(a)+\eta(a)]e(a,t) = k_1(a)s(a,t)\lambda(t), \quad a>0, t>0, \\ \dfrac{\partial i(a,t)}{\partial a} + \dfrac{\partial i(a,t)}{\partial t} + [\mu(a)+\gamma(a)]i(a,t) = \eta(a)e(a,t), \quad a>0, t>0, \\ \dfrac{\partial r(a,t)}{\partial a} + \dfrac{\partial r(a,t)}{\partial t} + \mu(a)r(a,t) = \nu(a)s(a,t) + \gamma(a)i(a,t), \quad a>0, t>0, \\ s(0,t) = B, \quad t \geq 0, \\ e(0,t) = i(0,t) = r(0,t) = 0, \quad t \geq 0, \\ \lambda(t) = \displaystyle\int_0^\infty k_2(a)i(a,t)da, \quad t \geq 0. \end{cases} \tag{2.1}$$

We note that problem (2.1) is an SEIR age-structured epidemic model that has been partly analyzed in Thieme (2001), where the existence of a unique endemic equilibrium is determined, and conditions for uniform weak disease persistence and disease extinction are derived.

In what follows, we determine the steady states proving a threshold theorem and obtain an explicitly computable threshold parameter R_ν, known as the reproduction number in the presence of a vaccination strategy ν as in Hadeler, et al. (1996) or the net replacement ratio as in Thieme (2001). If $R_\nu \leq 1$, then the only steady state is the disease-free equilibrium, and we prove that this steady state is globally stable if $R_\nu < 1$. If $R_\nu > 1$, then a disease-free equilibrium as well as an endemic equilibrium are possible steady states, we prove that the disease-free equilibrium is unstable if $R_\nu > 1$, and under suitable conditions, we prove that the endemic equilibrium is locally asymptotically stable, whenever it exists. Also, in some special cases, we prove that the endemic equilibrium is globally stable.

3 Reduction of the Model

In this section, we develop some preliminary formal analysis of problem (2.1). We define $p(a,t)$ by

$$p(a,t) = s(a,t) + e(a,t) + i(a,t) + r(a,t).$$

Then from (2.1), by adding the equations, we find that $p(a,t)$ satisfies the following:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial a} + \dfrac{\partial p(a,t)}{\partial t} + \mu(a)p(a,t) = 0, \quad a > 0, t > 0, \\ p(0,t) = B, \quad t \geq 0, \\ p(a,0) = p_0(a) = s_0(a) + e_0(a) + i_0(a) + r_0(a), \quad a \geq 0. \end{cases} \tag{3.1}$$

We note that problem (3.1) is of McKendrick-Von Foerster type, therefore it has a unique solution that exists for all time, see Bellman, et al. (1963), Hoppensteadt (1975) and Feller (1941). The unique solution of problem (3.1) is given by

$$p(a,t) = \begin{cases} p_0(a-t)\pi(a)/\pi(a-t), \quad a > t, \\ B\pi(a), \quad a < t, \end{cases} \tag{3.2}$$

where $\pi(a)$ is defined as

$$\pi(a) = e^{-\int_0^a \mu(\tau)d\tau}.$$

Also, from (2.1), $s(a,t), e(a,t), i(a,t)$ and $r(a,t)$ satisfy the following systems of equations:

$$\begin{cases} \dfrac{\partial s(a,t)}{\partial a} + \dfrac{\partial s(a,t)}{\partial t} + [\mu(a) + \nu(a)]s(a,t) = -k_1(a)s(a,t)\lambda(t), \quad a > 0, t > 0, \\ s(0,t) = B, \quad t \geq 0, \\ s(a,0) = s_0(a), \quad a \geq 0. \end{cases} \tag{3.3}$$

$$\begin{cases} \dfrac{\partial e(a,t)}{\partial a} + \dfrac{\partial e(a,t)}{\partial t} + [\mu(a) + \eta(a)]e(a,t) = k_1(a)s(a,t)\lambda(t), \quad a > 0, t > 0, \\ e(0,t) = 0, \quad t \geq 0, \\ e(a,0) = e_0(a), \quad a \geq 0. \end{cases} \tag{3.4}$$

$$\begin{cases} \dfrac{\partial i(a,t)}{\partial a} + \dfrac{\partial i(a,t)}{\partial t} + [\mu(a) + \gamma(a)]i(a,t) = \eta(a)e(a,t), \quad a > 0, t > 0, \\ i(0,t) = 0, \quad t \geq 0, \\ i(a,0) = i_0(a), \quad a \geq 0. \end{cases} \tag{3.5}$$

$$r(a,t) = p(a,t) - [s(a,t) + e(a,t) + i(a,t)]. \tag{3.6}$$

So, it is clear that (3.2)-(3.6) are equivalent to the original formulation (2.1).

4 The Steady States

In this section, we look at the steady state solution of problem (2.1). A steady state $s^*(a), e^*(a),$
$i^*(a), \lambda^*$ must satisfy the following equations:

$$\begin{cases} \dfrac{ds^*(a)}{da} + [\mu(a) + \nu(a)]s^*(a) = -k_1(a)s^*(a)\lambda^*, \quad a > 0, \\ \\ s^*(0) = B. \end{cases} \tag{4.1}$$

$$\begin{cases} \dfrac{de^*(a)}{da} + [\mu(a) + \eta(a)]e^*(a) = k_1(a)s^*(a)\lambda^*, \quad a > 0, \\ \\ e^*(0) = 0. \end{cases} \tag{4.2}$$

$$\begin{cases} \dfrac{di^*(a)}{da} + [\mu(a) + \gamma(a)]i^*(a) = \eta(a)e^*(a), \quad a > 0, \\ \\ i^*(0) = 0. \end{cases} \tag{4.3}$$

$$\lambda^* = \int_0^\infty k_2(a)i^*(a)da. \tag{4.4}$$

Anticipating our future needs, we define the following threshold parameter R_ν by

$$R_\nu = B\int_0^\infty \int_0^a \int_0^\sigma k_2(a)k_1(c)\pi(a)e^{-\int_c^\sigma \eta(\tau)d\tau}\eta(\sigma)e^{-\int_\sigma^a \gamma(\tau)d\tau}e^{-\int_0^c \nu(\tau)d\tau}dcd\sigma da. \tag{4.5}$$

Here, we note that the quantity R_0 obtained by setting $\nu = 0$ in the formula for R_ν is usually called the basic reproduction number, and is interpreted as the expected number of secondary cases produced in a lifetime by an infectious individual in the absence of the disease. Also, note that $R_\nu < R_0$ and R_ν is a decreasing function of ν.

In the following theorem, we determine the steady state solutions of problem (2.1).

Theorem (4.1).

(1) If $R_\nu > 1$, then $\lambda^* = 0$ and $\lambda^* > 0$ are possible steady states. The steady state with $\lambda^* > 0$ is unique when it exists, and it satisfies the following:

$$1 = B\int_0^\infty \int_0^a \int_0^\sigma k_2(a)k_1(c)\pi(a)e^{-\int_c^\sigma \eta(\tau)d\tau}\eta(\sigma)e^{-\int_\sigma^a \gamma(\tau)d\tau}e^{-\int_0^c [\nu(\tau)+k_1(\tau)\lambda^*]d\tau}dcd\sigma da. \tag{4.6}$$

And in this case $s^*(a), e^*(a), i^*(a)$ and $r^*(a)$ are given by

$$s^*(a) = B\pi(a)e^{-\int_0^a [\nu(\tau)+k_1(\tau)\lambda^*]d\tau}. \tag{4.7}$$

$$e^*(a) = \lambda^* B\pi(a)\int_0^a k_1(\sigma)e^{-\int_\sigma^a \eta(\tau)d\tau}e^{-\int_0^\sigma [\nu(\tau)+k_1(\tau)\lambda^*]d\tau}d\sigma. \tag{4.8}$$

$$i^*(a) = \lambda^* B\pi(a)\int_0^a \int_0^\sigma \eta(\sigma)k_1(c)e^{-\int_c^\sigma \eta(\tau)d\tau}e^{-\int_0^c [\nu(\tau)+k_1(\tau)\lambda^*]d\tau}e^{-\int_\sigma^a \gamma(\tau)d\tau}dcd\sigma. \tag{4.9}$$

$$r^*(a) = B\pi(a) - [s^*(a) + e^*(a) + i^*(a)]. \tag{4.10}$$

If $\lambda^* = 0$, we obtain the disease-free equilibrium given by

$$s^*(a) = B\pi(a)e^{-\int_0^a \nu(\tau)d\tau}, e^*(a) = i^*(a) = 0, r^*(a) = B\pi(a)\left[1 - e^{-\int_0^a \nu(\tau)d\tau}\right].$$

(2) If $R_\nu \leq 1$, then $\lambda^* = 0$ (the disease-free equilibrium) is the only steady state.

Proof. By solving (4.1), and substituting it in (4.2) and then solving (4.2), and substituting it in (4.3) and then solving (4.3), we obtain that $s^*(a), e^*(a)$ and $i^*(a)$ are given by (4.7), (4.8) and (4.9), respectively. From (3.2) and (3.6), we obtain that $r^*(a)$ satisfies (4.10).

From (4.4) and (4.9), we find that if $\lambda^* \neq 0$, then λ^* satisfies (4.6).

One can check that the right-hand side of (4.6) is a decreasing function of λ^* and approaches zero as $\lambda^* \to \infty$. Accordingly (4.6) has a unique solution $\lambda^* > 0$ iff $R_\nu > 1$. And in this case $s^*(a), e^*(a), i^*(a)$ are given by (4.7), (4.8), (4.9) and (4.10), respectively.

Otherwise, if $\lambda^* = 0$, then from (4.7), (4.8), (4.9) and (4.10), we obtain that $s^*(a) = B\pi(a)e^{-\int_0^a \nu(\tau)d\tau}, e^*(a) = i^*(a) = 0$, and $r^*(a) = B\pi(a)\left[1 - e^{-\int_0^a \nu(\tau)d\tau}\right]$. This completes the proof of the theorem.

Here, we note that the above theorem asserts the existence of an epidemic disease if $R_\nu > 1$. So, in order to control the spread of the disease and prevent an epidemic outbreak, one needs to apply a vaccination strategy ν_c to reduce R_ν to a value equal to one. If ν_c is constant then we have a unique way of obtaining $R_{\nu_c} = 1$, but in general ν depends on age and ν_c can be chosen according to some constraint that reduces the cost of vaccination or in general to obtain what is called an optimal vaccination strategy, for example, see Müller (1994), (1998), Hadeler, el al. (1996) and Castillo-Chavez, et al. (1998).

The effects of certain vaccination strategies for the eradication of important communicable diseases such as Measles, Pertussis, and Rubella are dealt with in several papers, for example, see Hethcote (1983), (1989), (1997), (2000), Dietz (1981), Katzmann, et al. (1984), Knox (1980), McLean (1986), Schenzle (1984), Coutinho, et al. (1993), Anderson, et al. (1999), and Greenhalgh (1990).

5 Stability of the Steady State

In this section, we study the stability of the steady states for problem (2.1) given by theorem (4.1).

From (3.2), we note that the total population has its steady state distribution $p_\infty(a) = B\pi(a)$, and from (3.1), $p_\infty(a)$ also satisfies the following:

$$\frac{dp_\infty(a)}{da} + \mu(a)p_\infty(a) = 0. \tag{5.1}$$

Now, we consider the following transformations, called the age profiles of susceptible, exposed, infective and immune individuals, respectively.

$$u(a,t) = \frac{s(a,t)}{p_\infty(a)}, \quad w(a,t) = \frac{e(a,t)}{p_\infty(a)}, \quad z(a,t) = \frac{i(a,t)}{p_\infty(a)}, \quad n(a,t) = \frac{r(a,t)}{p_\infty(a)}.$$

Then with these transformations, problem (2.1) becomes

$$\begin{cases} \dfrac{\partial u(a,t)}{\partial a}+\dfrac{\partial u(a,t)}{\partial t}+[\nu(a)+k_1(a)\lambda(t)]u(a,t)=0, \quad a>0,t>0, \\ \dfrac{\partial w(a,t)}{\partial a}+\dfrac{\partial w(a,t)}{\partial t}+\eta(a)w(a,t)=k_1(a)u(a,t)\lambda(t), \quad a>0,t>0, \\ \dfrac{\partial z(a,t)}{\partial a}+\dfrac{\partial z(a,t)}{\partial t}+\gamma(a)z(a,t)=\eta(a)w(a,t), \quad a>0,t>0, \\ \dfrac{\partial n(a,t)}{\partial a}+\dfrac{\partial n(a,t)}{\partial t}=\nu(a)u(a,t)+\gamma(a)z(a,t), \quad a>0,t>0, \\ u(0,t)=1, w(0,t)=z(0,t)=n(0,t)=0, \quad t\geq 0, \\ \lambda(t)=B\displaystyle\int_0^\infty k_2(a)z(a,t)\pi(a)da, \quad t\geq 0, \\ u(a,0)=u_0(a), w(a,0)=w_0(a), z(a,0)=z_0(a), n(a,0)=n_0(a), \quad a\geq 0. \end{cases} \tag{5.2}$$

From (5.2), $u(a,t), w(a,t)$ and $z(a,t)$ satisfy the following systems of equations:

$$\begin{cases} \dfrac{\partial u(a,t)}{\partial a}+\dfrac{\partial u(a,t)}{\partial t}+[\nu(a)+k_1(a)\lambda(t)]u(a,t)=0, \quad a>0,t>0, \\ u(0,t)=1, \quad t\geq 0, \\ u(a,0)=u_0(a)=s_0(a)/B\pi(a), \quad a\geq 0. \end{cases} \tag{5.3}$$

$$\begin{cases} \dfrac{\partial w(a,t)}{\partial a}+\dfrac{\partial w(a,t)}{\partial t}+\eta(a)w(a,t)=k_1(a)u(a,t)\lambda(t)], \quad a>0,t>0, \\ w(0,t)=0, \quad t\geq 0, \\ w(a,0)=w_0(a)=e_0(a)/B\pi(a), \quad a\geq 0. \end{cases} \tag{5.4}$$

$$\begin{cases} \dfrac{\partial z(a,t)}{\partial a}+\dfrac{\partial z(a,t)}{\partial t}+\gamma(a)z(a,t)=\eta(a)w(a,t), \quad a>0,t>0, \\ z(0,t)=0, \quad t\geq 0, \\ z(a,0)=z_0(a)=i_0(a)/B\pi(a), \quad a\geq 0. \end{cases} \tag{5.5}$$

By integrating problem (5.3) along characteristic lines $t-a=const.$, we find that $u(a,t)$ satisfies:

$$u(a,t)=\begin{cases} u_0(a-t)e^{-\int_0^t[\nu(a-t+\sigma)+k_1(a-t+\sigma)\lambda(\sigma)]d\sigma}, & a>t, \\ e^{-\int_0^a[\nu(\sigma)+k_1(\sigma)\lambda(t-a+\sigma)]d\sigma}, & a<t. \end{cases} \tag{5.6}$$

By integrating problem (5.4) along characteristic lines $t-a=const.$, we find that $w(a,t)$ satisfies:

$$w(a,t)=\begin{cases} w_0(a-t)e^{-\int_0^t \eta(a-t+\tau)d\tau}+\int_0^t e^{-\int_\sigma^t \eta(a-t+\tau)d\tau}k_1(a-t+\sigma)u(a-t+\sigma,\sigma)\lambda(\sigma)d\sigma, & a>t,\\ \int_0^a e^{-\int_\sigma^a \eta(\tau)d\tau}k_1(\sigma)u(\sigma,t-a+\sigma)\lambda(t-a+\sigma)d\sigma, & a<t. \end{cases} \tag{5.7}$$

By integrating problem (5.5) along characteristic lines $t-a=const.$, we find that $z(a,t)$ satisfies:

$$z(a,t)=\begin{cases} z_0(a-t)e^{-\int_0^t \gamma(a-t+\tau)d\tau}+\int_0^t e^{-\int_\sigma^t \gamma(a-t+\tau)d\tau}\eta(a-t+\sigma)w(a-t+\sigma,\sigma)d\sigma, & a>t,\\ \int_0^a e^{-\int_\sigma^a \gamma(\tau)d\tau}\eta(\sigma)w(\sigma,t-a+\sigma)d\sigma, & a<t. \end{cases} \tag{5.8}$$

By substituting (5.6) in (5.7) and then substituting the resultant in (5.8), we obtain that $z(a,t)$ satisfies the following:

$$z(a,t)=\begin{cases} z_0(a-t)e^{-\int_0^t \gamma(a-t+\tau)d\tau}+\int_0^t e^{-\int_\sigma^t \gamma(a-t+\tau)d\tau}\eta(a-t+\sigma)\Big\{w_0(a-t)e^{-\int_0^\sigma \eta(a-t+\tau)d\tau}\\ +u_0(a-t)\int_0^\sigma e^{-\int_c^\sigma \eta(a-t+\tau)d\tau}k_1(a-t+c)e^{-\int_0^c[\nu(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau}\lambda(c)dc\Big\}d\sigma, \ a>t,\\ \int_0^a\int_0^\sigma e^{-\int_\sigma^a \gamma(\tau)d\tau}\eta(\sigma)e^{-\int_c^\sigma \eta(\tau)d\tau}k_1(c)e^{-\int_0^c[\nu(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau}\lambda(t-a+c)dcd\sigma, \ a<t. \end{cases} \tag{5.9}$$

From (5.2), $\lambda(t)=B\int_0^\infty k_2(a)z(a,t)\pi(a)da$, then using (5.9), we find that $\lambda(t)$ satisfies the following:

$$\begin{aligned}\lambda(t)&=B\int_0^t\int_0^a\int_0^\sigma e^{-\int_\sigma^a \gamma(\tau)d\tau}\eta(\sigma)e^{-\int_c^\sigma \eta(\tau)d\tau}e^{-\int_0^c[\nu(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau}k_1(c)k_2(a)\pi(a)\lambda(t-a+c)dcd\sigma da\\ &+B\int_t^\infty k_2(a)\pi(a)z_0(a-t)e^{-\int_0^t \gamma(a-t+\tau)d\tau}da\\ &+B\int_t^\infty k_2(a)\pi(a)\int_0^t e^{-\int_\sigma^t \gamma(a-t+\tau)d\tau}\eta(a-t+\sigma)\Big\{w_0(a-t)e^{-\int_0^\sigma \eta(a-t+\tau)d\tau}\\ &+\int_0^\sigma e^{-\int_c^\sigma \eta(a-t+\tau)d\tau}k_1(a-t+c)u_0(a-t)e^{-\int_0^c[\nu(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau}\lambda(c)dc\Big\}d\sigma da. \end{aligned} \tag{5.10}$$

We note that by assumptions (2), (3), (5) and (8) of section 2 and the dominated

convergence theorem, we see that

$$\int_t^\infty k_2(a)\pi(a)z_0(a-t)e^{-\int_0^t \gamma(a-t+\tau)d\tau}da \to 0, \quad \text{as} \quad t \to \infty.$$

Also, by similar reasoning as above, we find that

$$\int_t^\infty k_2(a)\pi(a)\int_0^t e^{-\int_\sigma^t \gamma(a-t+\tau)\eta(a-t+\sigma)d\tau}\Bigg\{w_0(a-t)e^{-\int_0^\sigma \eta(a-t+\tau)d\tau}$$

$$+\int_0^\sigma e^{-\int_c^\sigma \eta(a-t+\tau)d\tau}k_1(a-t+c)u_0(a-t)e^{-\int_0^c[\nu(a-t+\tau)+k_1(a-t+\tau)\lambda(\tau)]d\tau}\lambda(c)dc\Bigg\}d\sigma da \to 0,$$

as $\quad t \to \infty.$

Consequently, from (5.10), $\lambda(t)$ has the following limiting equation (also, see Busenberg, et al. (1988)):

$$\lambda(t) = B\int_0^\infty\int_0^a\int_0^\sigma e^{-\int_0^c[\nu(\tau)+k_1(\tau)\lambda(t-a+\tau)]d\tau}\lambda(t-a+c)\eta(\sigma)k_1(c)e^{-\int_\sigma^a \gamma(\tau)d\tau}k_2(a)\pi(a)\times e^{-\int_c^\sigma \eta(\tau)d\tau}dcd\sigma da. \tag{5.11}$$

Now, we linearize the limiting equation (5.11) by considering perturbation $\xi(t)$ defined by

$$\xi(t) = \lambda(t) - \lambda^*.$$

If we define $K(c)$ by

$$K(c) = B\Bigg[\int_c^\infty\int_{a-c}^a e^{-\int_0^{a-c}[\nu(\tau)+k_1(\tau)\lambda^*]d\tau}\eta(\sigma)k_1(a-c)k_2(a)\pi(a)e^{-\int_\sigma^a \gamma(\tau)d\tau}e^{-\int_{a-c}^\sigma \eta(\tau)d\tau}d\sigma da$$
$$-\lambda^*\int_c^\infty\int_{a-c}^a\int_{a-c}^\sigma e^{-\int_0^\tau[\nu(s)+k_1(s)\lambda^*]ds}k_1(a-c)k_1(\tau)k_2(a)\pi(a)\eta(\sigma)e^{-\int_\sigma^a \gamma(s)ds}e^{-\int_\tau^\sigma \eta(s)ds}d\tau d\sigma da\Bigg], \tag{5.12}$$

then the linearization of the limiting equation (5.11) can be rewritten as

$$\xi(t) = \int_0^\infty K(c)\xi(t-c)dc. \tag{5.13}$$

The characteristic equation for (5.13) is given by

$$\hat{K}(s) = 1, \tag{5.14}$$

where $\hat{K}(s) = \int_0^\infty e^{-sc}K(c)dc.$

In the following theorem, we show that the disease-free equilibrium, $\lambda^* = 0$, is unstable if $R_\nu > 1$, and locally asymptotically stable if $R_\nu < 1$.

Theorem (5.1). The disease-free equilibrium, $\lambda^* = 0$, is unstable if $R_\nu > 1$, and locally

asymptotically stable if $R_\nu < 1$.

Proof. We note that if $\lambda^* = 0$, then from (5.12), $K(c)$ satisfies the following:

$$K(c) = B\int_c^\infty \int_{a-c}^a e^{-\int_0^{a-c}\nu(\tau)d\tau}\eta(\sigma)k_1(a-c)k_2(a)\pi(a)e^{-\int_\sigma^a \gamma(\tau)d\tau}e^{-\int_{a-c}^\sigma \eta(\tau)d\tau}d\sigma da.$$

Changing the order of integration several times and making appropriate changes of variables yields

$$\int_0^\infty K(c)dc = R_\nu.$$

In (5.14), if we take $s = x$, where x is real, and noticed that $\int_0^\infty e^{-xc}K(c)dc$ is a decreasing function for $x > 0$ and has a value $R_\nu > 1$ for $x = 0$, and approaches zero as $x \to \infty$, accordingly $\exists\ x^* > 0$ such that the characteristic equation (5.14) is satisfied, and therefore the disease-free equilibrium, $\lambda^* = 0$, is unstable if $R_\nu > 1$.

If $R_\nu < 1$, we note that the characteristic equation (5.14) will not be satisfied for any s with $Re\ s \geq 0$ because $K(c)$ is non-negative and therefore,

$$|\hat{K}(s)| \leq \int_0^\infty e^{-(Re\ s)c}K(c)dc \leq \int_0^\infty K(c)dc = R_\nu < 1,$$

therefore, the characteristic equation (5.14) will not be satisfied for any s with $Re\ s \geq 0$. Hence the disease-free equilibrium, $\lambda^* = 0$, is locally asymptotically stable if $R_\nu < 1$. This completes the proof of the theorem.

In the next result, we show that the disease-free equilibrium is globally stable when $R_\nu < 1$.

Theorem (5.2). Suppose that $R_\nu < 1$. Then the disease-free equilibrium is globally stable.

Proof. Let $\lambda^\infty = \limsup_{t\to\infty} \lambda(t)$, then by using the limiting equation (5.11) and $Fatou's$ Lemma, we obtain the following:

$$\begin{aligned}\lambda^\infty &\leq \lambda^\infty B\int_0^\infty\int_0^a\int_0^\sigma e^{-\int_0^c \nu(\tau)d\tau}\eta(\sigma)e^{-\int_c^\sigma \eta(\tau)d\tau}k_1(c)e^{-\int_\sigma^a \gamma(\tau)d\tau}k_2(a)\pi(a)dcd\sigma da\\ &= \lambda^\infty R_\nu < \lambda^\infty,\end{aligned}$$

which gives $\lambda^\infty = 0$. That is, the disease-free equilibrium is globally stable. This completes the proof of the theorem.

In order to study the stability of the endemic equilibrium, $\lambda^* > 0$, we need to show that the kernel $K(c)$ is non-negative, therefore, we impose the following condition:

$$\lambda^*\int_0^\infty e^{-\int_0^\tau[\lambda^* k_1(s)-\eta(s)]ds}k_1(\tau)d\tau < 1. \tag{5.15}$$

To see how condition (5.15) would imply that $K(c) \geq 0$, we shall prove the following lemma.

Lemma (5.1). Suppose that (5.15) holds, then $g(x) = \lambda^* \int_x^D e^{-\int_x^\tau [\lambda^* k_1(s) - \eta(s)]ds} k_1(\tau)d\tau < 1$,
$\forall\ x \in [0, D]$, where D is any non-negative real number.

Proof. Observe that by (5.15), $g(0) < 1$, and by definition, $g(D) = 0$. Also, note that $g'(x) = -\lambda^* k_1(x) - [\eta(x) - \lambda^* k_1(x)]g(x)$. Thus, if we assume that $g(x) < 1$, then

$$g'(x) \leq \begin{cases} -\lambda^* k_1(x) & \text{if} \quad [\eta(x) - \lambda^* k_1(x)] \geq 0, \\ -\eta(x), & \text{if} \quad [\eta(x) - \lambda^* k_1(x)] < 0. \end{cases}$$

Therefore $g'(x) \leq 0$, provided that $g(x) < 1$. Since $g(0) < 1$, this implies that $g(x) < 1,\ \forall x \in [0, D]$. This completes the proof of the lemma.

From Lemma (5.1), we deduce that $K(c)$ is non-negative.

In the next result, we show that the endemic equilibrium, $\lambda^* > 0$, is locally asymptotically stable when $R_\nu > 1$ and condition (5.15) holds.

Theorem (5.3). Suppose that

(1) $R_\nu > 1$,

(2) condition (5.15) is satisfied.

Then the endemic equilibrium, $\lambda^* > 0$, is locally asymptotically stable.

Proof. For the characteristic equation $\hat{K}(s) = 1$, suppose that $Re\ s \geq 0$ then $|\hat{K}(s)| \leq \int_0^\infty e^{(-Re\ s)c}\ K(c)dc \leq \int_0^\infty K(c)dc < 1$, note that the first inequality because K is non-negative by assumption 2, and the second inequality because $Re\ s \geq 0$ and the last inequality because of assumption 2, Lemma (5.1) and equation (4.6). Therefore, the characteristic equation cannot be satisfied for any s with $Re\ s \geq 0$, i.e. the endemic equilibrium is locally asymptotical stable, whenever it exists, provided that condition (2) is satisfied. This completes the proof of the theorem.

In the next result, we prove the global stability of the endemic equilibrium under a suitable condition.

Theorem (5.4). Suppose that $\nu(a) \equiv \eta(a)$. Then the endemic equilibrium is globally stable.

Proof. Using the limiting equation (5.11), we obtain that $\lambda(t)$ satisfies the following:

$$\lambda(t) \ = \ B \int_0^\infty \int_0^a k_2(a)\pi(a)\eta(\sigma) e^{-\int_0^\sigma \eta(\tau)d\tau} e^{-\int_\sigma^a \gamma(\tau)d\tau} \left[1 - e^{-\int_0^\sigma k_1(\tau)\lambda(t-a+\tau)d\tau}\right] d\sigma da.$$

Now, letting $v(t) = \lambda(t) - \lambda^*$, we obtain the following:

$$v(t) \ = \ B \int_0^\infty \int_0^a k_2(a)\pi(a)\eta(\sigma) e^{-\int_0^\sigma \eta(\tau)d\tau} e^{-\int_\sigma^a \gamma(\tau)d\tau} e^{-\lambda^* \int_0^\sigma k_1(\tau)d\tau} \left[1 - e^{-\int_0^\sigma k_1(\tau)v(t-a+\tau)d\tau}\right] d\sigma da. \tag{5.16}$$

Now, If we use the fact that $1 - e^{-\int_0^\sigma k_1(\tau)v(t-a+\tau)d\tau} \leq \int_0^\sigma k_1(\tau)v(t-a+\tau)d\tau$ in (5.16), and then use $Fatou's$ Lemma and equation (4.6), we obtain that $\underset{t\to\infty}{limsup}\,|v(t)| = 0$. Therefore, the endemic equilibrium is globally stable. This completes the proof of the theorem.

6 Conclusion

We studied an SEIR age-structured epidemic model when susceptible and immune individuals are vaccinated indiscriminately and assumed proportionate mixing for the force of infection. The importance of this work stems from the fact that, to the best of our knowledge, Thieme (2001) and Li, et al. (2001), are the only two papers in the literature that have dealt with SEIR type epidemic models with age-structure.

We determined the steady states of the model and examined their stability by determining a computable threshold parameter R_ν, usually known as the reproduction number in the presence of the vaccination strategy $\nu(a)$ or the net replacement ratio. R_ν decreases with $\nu(a)$ and is used to determine a critical vaccination coverage which will eradicate the disease with minimum vaccination coverage.

If $R_\nu \leq 1$, then the only steady state is the disease-free equilibrium and is globally stable, if $R_\nu < 1$. If $R_\nu > 1$, then a disease-free equilibrium as well as an endemic equilibrium (unique when it exists) are possible steady states, the disease-free equilibrium is unstable and the endemic equilibrium is locally asymptotically stable, if condition (5.15) is satisfied. Furthermore, if $\nu(a) \equiv \eta(a)$, then the endemic equilibrium is globally stable.

We note that for an SIR age-structured epidemic model with proportionate mixing for the force of infection, Thieme (1990), showed that under certain conditions, the endemic equilibrium could undergo stability change. This may be the case here as well.

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